Combinatorics, 2020 Fall, USTC
Homework 1

- The due is on Tuesday, Sep. 22.

1. Let n, r be positive integers and $n \geq r$. Give a **combinatorial proof** of

$$\binom{r}{r} + \binom{r+1}{r} + ... + \binom{n}{r} = \binom{n+1}{r+1}.$$

2. Let n be a positive integer. Prove that the identity

$$x^n = \sum_{k=1}^{n} S(n,k)(x)_k$$

holds for every real number x, where $S(n,k)$ is the Strirling number of the second kind, and $(x)_k := x(x-1)...(x-k+1)$ denotes a polynomial of degree k with variable x.

Hint: first prove the case when x is a positive integer by double-counting certain mappings.

3. Let n, r be integers satisfying $0 \leq r \leq 2n$. Find the value of $\sum_{i=0}^{n}(-1)^i\binom{n}{i}\binom{n}{r-i}$.

4. For any integer $n \geq 2$, let $\pi(n)$ be the number of primes in $\{1, 2, ..., n\}$.

(a) Prove that the product of all primes p satisfying $m < p \leq 2m$ is at most $\binom{2m}{m}$, where $m \geq 1$ is any integer.

(b) Use (a) to prove that $\pi(n) \leq \frac{Cn}{\log n}$ for some absolute constant C. (Hint: by induction and use the estimation on $\binom{2m}{m}$)

5. How many ways are there to seat n couples at a round table with $2n$ chairs in such a way that none of the couples sit next to each other? If one seating plan can be obtained from other plan by a rotation, then we will view them as one plan.

6. Prove the following statements.

(a). If p is odd, then $|A_1 \cup A_2 \cup ... \cup A_n| \leq \sum_{k=1}^{p}(-1)^{k+1}S_k$;

(b). If p is even, then $|A_1 \cup A_2 \cup ... \cup A_n| \geq \sum_{k=1}^{p}(-1)^{k+1}S_k$.

Here, S_k is the sum of the sizes of all k-fold intersections as defined in class.