Combinatorics, 2020 Fall, USTC
Homework 10

- The due is on **Tuesday**, Dec. 22, at beginning of the class.

1. Prove that for all integers n and $p \in [0,1]$,

$$R(k,\ell) > n - \binom{n}{k} p^{\binom{k}{2}} - \binom{n}{\ell}(1-p)^{\binom{\ell}{2}}$$

and then show

$$R(4,k) \geq c \cdot \left(\frac{k}{\ln k}\right)^2$$

for some absolute constant $c > 0$.

2. Let $\mathcal{F} \subseteq \binom{[n]}{3}$ be a family of m subsets where $m \geq n/3$. Prove that there exists a subset $A \subseteq [n]$ with

$$|A| \geq \frac{2n^{3/2}}{3\sqrt{3m}}$$

such that none of the sets in $\mathcal{F}$ is contained in A.

3. Using the corollary proved in the beginning of the lecture on Dec 15, prove the approximate version of Turán's Theorem: For any n-vertex K_{r+1}-free graph G, we have $e(G) \leq \frac{r-1}{2r} n^2$.

4. Prove the exact version of Turán's Theorem: For any n-vertex K_{r+1}-free graph G,

$$e(G) \leq e(T_r(n))$$

with equality if and only if G is the Turán graph $T_r(n)$. (Hint: by induction on r.)

5. Let $\mathcal{F}$ be an independent system of $[n]$. Let $\pi \in S_n$ be a random permutation of $[n]$. By considering the set

$$X = \{i : \{\pi(1), \pi(2), ..., \pi(i)\} \in \mathcal{F}\},$$

give a new proof of that

$$|\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor}.$$