Combinatorics, 2020 Fall, USTC
Homework 2

- The due is on Saturday, October 10, at beginning of the class.

1. Prove the following statements.

(a). If p is odd, then $|A_1 \cup A_2 \cup ... \cup A_n| \leq \sum_{k=1}^{p}(-1)^{k+1}S_k$;

(b). If p is even, then $|A_1 \cup A_2 \cup ... \cup A_n| \geq \sum_{k=1}^{p}(-1)^{k+1}S_k$.

Here, S_k is the sum of the sizes of all k-fold intersections as defined in class.

2. How many functions $f : [n] \to [n]$ are there such that for all $i < j$, we have $f(i) \leq f(j)$?

3. Find a closed formula of

$$\sum_{k=0}^{\lfloor n/7 \rfloor} \binom{n}{7k}.$$

(Try to simplify this expression to a summation with at most four terms.)

Hint: let $\epsilon = e^{2\pi i/7}$, where i is the imaginary unit, and then consider the value of $\sum_{j=0}^{6} \epsilon^{kj}$.

4. An *equivalence* of $[n]$ is a partition of $[n]$ into k non-ordered non-empty subsets for some k. Denote the n^{th} *Bell number* B_n to be the total number of equivalences of $[n]$. Prove that

$$B_n = \frac{1}{e} \sum_{i=0}^{\infty} \frac{i^n}{i!}.$$

5. (a). Given a natural number N, determine the probability that two numbers $m, n \in [N]$ chosen independently at random are relatively prime.

(b). Prove that the limit of the probability in (a) for $N \to \infty$ equals the infinite product $\prod_p (1 - 1/p^2)$, where p runs over all primes.

6. Show that the number $(6 + \sqrt{37})^{999}$ has at least 999 zero following the decimal point.

7. Let $f_n(x) = x(x-1)...(x-n+1)$. Prove that

$$f_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} f_k(x) f_{n-k}(y).$$

8. Let A_n be the number of ways of going up n stairs, if we may take one or two steps at a time. Find the generating function $f(x)$ of $\{A_n\}_{n \geq 0}$.

9. In how many ways can one divide a convex n-gon into triangles by $n-3$ non-intersecting diagonals in such a way that each triangle has an edge in common with the convex n-gon?