Combinatorics, 2020 Fall, USTC
Homework 3

- The due is on Tuesday, October 20, at beginning of the class.

1. Recall the Stirling number $S(n,k)$ of the second kind.

(1). Prove that for all $1 \le k \le n$,

$$S(n+1,k) = S(n,k-1) + k \cdot S(n,k).$$

(2). For a fixed k, let $A_k(x)$ be the ordinary generating function of the sequence $\{S(n,k)\}_{n\ge 0}$. Find a closed form of $A_k(x)$. (When $0 \le n < k$, $S(n,k)$ is defined to be 0.)

2. Let a_n denote the number of mappings $f : [n] \to [n]$ such that if f takes a value i, then it also takes every value j for $1 \le j \le i$. Let $a_0 = 1$. Find the closed form of the generating function $f(x)$ of $\{a_n\}_{n\ge 0}$.

3. A *binary tree* can be defined by induction as follows: a binary tree either is empty (it has no vertex), or consists of one distinguished vertex called the *root* plus an ordered pair of binary trees called the *left subtree* and *right subtree*. Let b_i be the number of binary trees with n vertices. So $b_0 = 1, b_1 = 1, b_2 = 2, b_3 = 5$ and so on. Find a closed form for b_n.

4. Consider a random walk along the x-axis, where we start at the integer 0 and in each coming step we move from integer i to $i+1$ or to $i-1$ with probability 1/2.

(1). Prove that every integer k is visited at least once with probability 1.

(2). For any positive integer $k \ge 1$, what is the expected number of steps needed to get at least k steps away from 0 (i.e., to reach k or $-k$)?

5. Given arrangements of a special type, called the type I. Let a_n be the number of such arrangements of n people, where $a_0 = 0$ (no empty group is allowed). Let $A(x)$ be the exponential generating function of $\{a_n\}$. We now define arrangements of n people of a new type, called the type II, as follows. Fix $k \ge 1$. An arrangement of n people of type II is obtained by dividing the given n people into k groups, called the 1^{st} group, the 2^{nd} group,..., and the k^{th} group, and arranging each group by an arrangement of type I. Let b_n be the number of arrangements of n people of type II; and let $B(x)$ be the exponential generating function of $\{b_n\}$. Express the exponential generating function $B(x)$ using $A(x)$ in a closed form.

6. Construct as many pairwise non-isomorphic graphs with vertex set $[n]$ as possible (suppose that n is very large number). Prove that why your graphs are non-isomorphic. Can you find more than n^2 of them? At least $2^{n/10}$? Or even at least $2^{n^2/10}$?

7. Let G be a graph in which all vertices have degree at least d. Prove that G contains a cycle of length at least $d+1$.