Combinatorics, 2020 Fall, USTC
Homework 4

- The due is on Tuesday, October 27, at beginning of the class.

1. Let G be a graph with maximum degree 3. Prove that its vertices can be colored by 2 colors (each vertex gets one color) in such a way that there is no path of length two whose 3 vertices all have the same color.

2. Prove that any graph G with an even number of vertices has two vertices with an even number of common neighbors.

3. Sperner's lemma in dimension 3.

(1). Consider a tetrahedron $T = A_1A_2A_3A_4$, where A_i is one of the four vertices, in the 3-dimensional space, and some subdivision of T into small tetrahedrons, such that each face of each small tetrahedron either lies on a face of the big tetrahedron or is also a face of another small tetrahedron. (Just imagine that the subdivision of T is as normal as you would divide into.) Let us label the vertices of the small tetrahedron by colors 1,2,3,4, in such a way that the vertex A_i get the color i, the edge A_iA_j only contains vertices colored by i and j, and the face $A_iA_jA_k$ has only colors i, j and k. Prove that there exists a small tetrahedron with colors 1,2,3,4.

(2). Formulate and prove a 3-dimensional version of Brouwer's fixed point theorem (about continuous mappings of a tetrahedron into itself).

4. A family $\mathcal{F}$ of $[n]$ is *intersecting*, if for any two subsets $A, B \in \mathcal{F}$, $A \cap B \neq \emptyset$. For any intersecting family $\mathcal{F}$ of $[n]$, prove that there exists an intersecting family $\mathcal{F}'$ of $[n]$ such that $\mathcal{F} \subseteq \mathcal{F}'$ and $|\mathcal{F}'| = 2^{n-1}$.

5. Let $n > 0$ be an even integer. Let $\mathcal{F} \subset 2^{[n]}$ be a family of subsets of $[n]$ such that $\mathcal{F}$ contains no four distinct sets A, B, C, D satisfying $A \subset B \subset C \subset D$. Show that $|\mathcal{F}| \leq 3\binom{n}{n/2}$.

6. Show that the set families $\binom{[n]}{\lfloor n/2 \rfloor}$ and $\binom{[n]}{\lceil n/2 \rceil}$ are the only independent systems in $2^{[n]}$ with the largest possible number of sets.