Combinatorics, 2020 Fall, USTC
Homework 5

- The due is on Tuesday, Nov. 3, at beginning of the class.

1. Let G be a graph such that for any two non-adjacent vertices u, v, it holds that $N_G(u) = N_G(v)$. Prove that G must be a *complete multipartite graph* (i.e., $V(G)$ can be partitioned into $V_1, V_2, ..., V_k$ for some $k \geq 2$ such that for any $u \in V_i, v \in V_j$ where $i \neq j$, we have $uv \in E(G)$).

2. Let L be a set of n distinct lines in the plane and P a set of n distinct points in the plane. Prove that the number of pairs (p, ℓ), where $p \in P$, $\ell \in L$, and p lies on ℓ, is bounded from above by $O(n^{3/2})$.

3. Show that if a graph G on n vertices does not contain $K_{s,t}$ as a subgraph, then it has at most $C \cdot n^{2-1/s}$ edges for some absolute constant C only depending on t and s.

4. Let $t_n = ST(K_n)$. Prove the following recurrent formula

$$(n-1)t_n = \sum_{k=1}^{n-1} k(n-k)\binom{n-1}{k-1} t_k t_{n-k}.$$

5. For all positive integers n, determine the number of spanning trees on given n vertices in which all vertices have degree 1, 2, or 3.

6. For each $1 \leq k \leq n-1$, let N_k denote the number of spanning trees of K_n in which the vertex n has degree k. (Here we assume that $V(K_n) = \{1, 2, ..., n\}$.)

(a) Prove that $(n-1-k)N_k = k(n-1)N_{k+1}$.

(b) Prove by induction on k that $N_k = \binom{n-2}{k-1}(n-1)^{n-1-k}$.

(c) Use the above equality to derive the Cayley's formula.