Combinatorics, 2020 Fall, USTC
Homework 6

- The due is on Tuesday, Nov. 10, at beginning of the class.

1. Given a tree T and two vertices x, y of T, let $d(x, y)$ be the length of the unique path in T between x and y. Determine those trees T on n vertices for which

$$\sum_{x,y \in V(T)} d(x, y)$$

is maximal and minimal, respectively.

2. Show that any tree has more leaves than vertices of degree at least 3.

3. Given a graph G on n vertices, let Q be its Laplace matrix, and let Q^* be the matrix whose element at position (i, j) equals $(-1)^{i+j} \det Q_{ij}$. Prove the following:

(1). $\det Q = 0$.

(2). If G is connected, then Q has rank $n - 1$.

(3). If G is disconnected, then the rank of Q is at most $n - 2$.

(4). If G is connected and $\vec{x} \in R^n$ is an arbitrary vector, then $Q\vec{x} = \vec{0}$ if and only if $\vec{x}$ is a multiple of the vector $\vec{1} = (1, 1, ..., 1)^T$.

(5). Show that QQ^* is a zero matrix. Using (4), infer that Q^* is an identical matrix.

4. Calculate $ST(K_{n,m})$, i.e., the number of spanning trees of the complete bipartite graph $K_{n,m}$.

5. Let $n = 2k$. Characterize all interesting families $\mathcal{F} \subseteq \binom{[n]}{k}$ with $|\mathcal{F}| = \binom{n-1}{k-1}$.

6. Let $n > 2k$ and $\mathcal{F}$ be an intersecting family in $\binom{[n]}{k}$ of size $\binom{n-1}{k-1}$. Following the notation given in class, prove that if $\mathcal{F}_\pi = \{A_1, A_2, ..., A_k\}$ for some cyclic permutation π, then $|A_1 \cap A_2 \cap ... \cap A_k| = 1$ and $|A_1 \cup A_2 \cup ... \cup A_k| = 2k - 1$.