Combinatorics, 2020 Fall, USTC
Homework 8

- The due is on Thursday, Nov. 26 (UNUSUAL date!), at beginning of the class.

1. For any point $p \in R^d$ in d-dimension, write $p = (p_1, p_2, ..., p_d)$. A set $\mathcal{P}$ of points in R^d is called *good*, if for each $i \in [d]$, the i^{th} coordinates of these points are distinct. Given two points $p, q \in R^d$, define $box(p, q) := \{x \in R^d : \min\{p_i, q_i\} \leq x_i \leq \max\{p_i, q_i\} \text{ for each } i\}$ as the box determined by points p, q.

Prove that in any good set $\mathcal{P}$ of $2^{2^{d-1}} + 1$ points of R^d, there is a point $x \in \mathcal{P}$ which is in the box determined by two of the other points in $\mathcal{P}$.

2. For any integers $k, \ell \geq 1$, construct a sequence of $k\ell$ distinct integers with no increasing subsequence of length $k + 1$ and with no decreasing subsequence of length $\ell + 1$.

3. Let $P = (X, \prec)$ be a finite partial ordered set. Show that X can be expressed as a disjoint union of at most $\alpha(P)$ chains.

Hint: by induction on $|X|$.

4. Construct an explicit 2-edge-coloring of $K_{k\ell}$ to show that $R(k + 1, \ell + 1) \geq k\ell + 1$.

5. Use Ramsey's theorem to prove: for every integer $k \geq 2$, there is an integer n such that every sequence of n distinct real numbers contains a monotone subsequence of k real numbers.

(You must use Ramsey's theorem and cannot use the Erdős-Szekeres theorem.)

6. For integers $k \geq 2$ and $s_1, s_2, ..., s_k \geq 2$, the *Ramsey number* $R_k(s_1, s_2, ..., s_k)$ is the least number of integer n such that any k-edge-coloring of K_n has a monochromatic clique K_{s_i} in color i. Prove that for any integers $s_1, s_2, ..., s_k \geq 2$, the Ramsey Number

$$R_k(s_1, s_2, ..., s_k) < +\infty.$$

7. Prove that $2^k \leq R_k(3, 3, ..., 3) \leq (k + 1)!$