Combinatorics, 2020 Fall, USTC
Homework 9

- The due is on **Tuesday**, Dec. 8, at beginning of the class.

1. Let V be a finite set and $r \geq 3$ be an integer. An *r-uniform hypergraph* is a pair (V, E), where $E \subseteq \binom{V}{r}$. And $K_n^{(r)}$ denotes the complete r-uniform hypergraph on n vertices, i.e., $K_n^{(r)} = (V, \binom{V}{r})$ for $|V| = n$. Let *hypergraph Ramsey number* $R^{(r)}(s,t)$ be the least integer n such that any 2-edge-coloring of $K_n^{(r)}$ has a blue $K_s^{(r)}$ or a red $K_t^{(r)}$.

Prove that for any $s, t \geq r$, $R^{(r)}(s,t) < \infty$.

2. (a). A fixed point of a permutation π is an integer i such that $\pi(i) = i$. Let X_π be the number of fixed points of permutation π. What is the expectation $E[X_\pi]$ for a random permutation π chosen from the set S_n? (Recall that S_n is the set of all permutations of $[n]$.)

(b). We toss a fair coin n times. What is the expected number of "runs"? "Runs" are consecutive tosses with the same result. For example, the sequence "TTTHHHHTT" has 3 runs, where 'T' means 'tail' and 'H' means 'Head'.

3. Let $n \geq 6$. We color the edges of complete graph K_n with colors blue and red in such a way that each edge is in at most one blue triangle. Let X be a maximal subset of the vertices with the property that X does not contain a blue triangle. Then

(a) show that X contains at least $|X|/2$ red edges, and

(b) show that $|X| \geq \sqrt{2n}$.

4. Let $k \geq 4$ be an integer and $\mathcal{F} \subseteq \binom{X}{k}$ be a k-family defined on the ground set X. Prove that if

$$|\mathcal{F}| < \frac{4^{k-1}}{3^k},$$

then there is a coloring of elements of X with 4 colors such that all 4 colors are represented in every subset of family $\mathcal{F}$.