Combinatorics

Instructor: Jie Ma, Scribed by Jun Gao, Jialin He and Tianchi Yang

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1 Part I. Enumeration

First we give some standard notation that will be used throughout this course. Let n be a positive integer. We will use $[n]$ to denote the set $\{1, 2, ..., n\}$. Given a set X, $|X|$ denotes the number of elements contained in X. Sometimes we also use "#" to express the word "number". The *factorial* of n is the product

$$n! = n \cdot (n-1) \cdots 2 \cdot 1,$$

which can be extended to all non-negative integers by letting $0! = 1$.

1.1 Binomial coefficients

Let X be a set of size n. Define $2^X = \{A : A \subseteq X\}$ to be the family of all subsets of X. So $|2^X| = 2^{|X|} = 2^n$. Let $\binom{X}{k} = \{A : A \subseteq X, |A| = k\}$.

Fact 1.1. *For integers $n > 0$ and $0 \le k \le n$, we have $|\binom{X}{k}| = \binom{n}{k} = \frac{n!}{k!(n-k)!}$.*

Proof. If $k = 0$, then it is clear that $|\binom{X}{0}| = |\{\emptyset\}| = 1 = \binom{n}{0}$. Now we consider $k > 0$. Let

$$(n)_k := n(n-1)\cdots(n-k+1) = \frac{n!}{(n-k)!}.$$

First we will show that number of order k-tuples $(x_1, x_2, ..., x_k)$ with distinct $x_i \in X$ is $(n)_k$. There are n choices for the first element x_1. When $x_1, ..x_i$ is chosen, there are exactly $n-i$ choices for the element x_{i+1}. So the number of order k-tuples $(x_1, x_2, ..., x_k)$ with distinct $x_i \in X$ is $(n)_k$. Since any subset $A \in \binom{X}{k}$ correspond to $k!$ ordered k-tuples, it follows that $|\binom{X}{k}| = \frac{(n)_k}{k!} = \frac{n!}{k!(n-k)!}$. This finishes the proof. ∎

Next we discuss more properties of binomial coefficients. For positive integers n strictly less than k, we let $\binom{n}{k} = 0$.

Fact 1.2. (1). $\binom{n}{k} = \binom{n}{n-k}$ *for $0 \le k \le n$.*
(2). $2^n = \sum_{0 \le k \le n} \binom{n}{k}$.
(3). $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.

Proof. (1) is trivial. Since $2^{[n]} = \cup_{0 \le k \le n} \binom{[n]}{k}$, we see $2^n = \sum_{0 \le k \le n} \binom{n}{k}$, proving (2). Finally, we consider (3). Note that the first term on the right hand side $\binom{n-1}{k-1}$ is the number of k-sets containing a fixed element, while the second term $\binom{n-1}{k}$ is the number of k-sets avoiding this element. So their summation gives the total number of k-sets in $[n]$, which is $\binom{n}{k}$. This finishes the proof. ∎

Pascal's triangle is a triangular array constructed by summing adjacent elements in preceding rows. By Fact 1.2 (3), in the following graph we have that the kth element in the n row is $\binom{n}{k-1}$.

$$
\begin{array}{ll}
1 \cdots & 1 \quad 1 \\
2 \cdots & 1 \quad 2 \quad 1 \\
3 \cdots & 1 \quad 3 \quad 3 \quad 1 \\
4 \cdots & 1 \quad 4 \quad 6 \quad 4 \quad 1 \\
5 \cdots & 1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
6 \cdots & 1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \\
7 \cdots & 1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1 \\
8 \cdots & 1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1 \\
9 \cdots & 1 \quad 9 \quad 36 \quad 84 \quad 126 \quad 126 \quad 84 \quad 36 \quad 9 \quad 1 \\
10 \cdots & 1 \quad 10 \quad 45 \quad 120 \quad 210 \quad 252 \quad 210 \quad 120 \quad 45 \quad 10 \quad 1
\end{array}
$$

Fact 1.3. *The number of integer solutions* $(x_1, ..., x_n)$ *to the equation* $x_1 + \cdots x_n = k$ *with each* $x_i \in \{0, 1\}$ *is* $\binom{n}{k}$.

Fact 1.4. *The number of integer solution* $(x_1, ...x_n)$ *with each* $x_i \geq 0$*, to the equation* $x_1 + \cdots x_n = k$ *is* $\binom{n+k-1}{n-1}$.

Proof. Suppose we have k sweets (of the same sort), which we want to distribute to n children. In how many ways can we do this? Let x_i denote the number of sweets we give to the i-th child, this question is equivalent to that state above.

We lay out the sweets in a single row of length r and let the first child pick them up from left to right (can be 0). After a while we stop him/her and let the second child pick up sweets, etc. The distribution is determined by the specifying the place of where to start a new child. Equal to select $n-1$ elements form $n+r-1$ elements to be the child, others be the sweets (the first child always starts at the beginning). So the answer is $\binom{n+k-1}{n-1}$ ∎

Exercise 1.1 Prove that

$$\sum_{k=0}^{m} \binom{m}{k} \binom{n+k}{m} = \sum_{k=0}^{m} \binom{n}{k} \binom{m}{k} 2^k.$$

1.2 Counting mappings

Define X^Y to be the set of all functions $f : Y \to X$.

Fact 1.5. $|X^Y| = |X|^{|Y|}$.

Proof. Let $|Y| = r$. We can view X^Y as the set of all strings $x_1 x_2 ... x_r$ with elements $x_i \in X$, indexed by the r element of Y. So $|X^Y| = |X|^{|Y|}$. ∎

Fact 1.6. *The number of injective functions* $f : [r] \to [n]$ *is* $(n)_r$.

Proof. We can view the injective function f as a order k-tuples $(x_1, x_2, ..., x_r)$ with distinct $x_i \in X$, so the number of injective functions $f : [r] \to [n]$ is $(n)_r$. ∎

Definition 1.7 (The Stirling number of the second kind). *Let S(r,n) be the number of partition of* $[r]$ *into* n *unordered non-empty parts.*

Exercise 1.2

$$S(r,2)=\frac{2^r-2}{2}=\frac{1}{2}\sum_{i=1}^{r-1}\binom{r}{i}.$$

Fact 1.8. *The number of surjective functions $f:[r]\to[n]$ is $n!S(r,n)$.*

Proof. Since f is a surjecture function $\iff \forall i\in[n], f^{-1}(i)\neq\emptyset \iff \cup_{i\in[n]}f^{-1}(i)=[r]$, and $S(r,n)$ is the number of partition of $[r]$ into n unordered non-empty parts, we have the number of surjective functions $f:[r]\to[n]$ is $n!S(r,n)$. ∎

We say that any injective $f:X\to X$ is a **permutation** of X (also a bijection). We may view a permutation in two ways: (1) it is a bijective from X to X. (2) a reordering of X.

Cycle notation describes the effect of repeatedly applying the permutation on the elements of the set. It expresses the permutation as a product of cycles; since distinct cycles are disjoint, this is referred to as "decomposition into disjoint cycles".

Definition 1.9. *The Stirling number of the first kind $s(r,n)$ is $(-1)^{(r-n)}$ times the number of permutations of $[r]$ with exactly n cycles.*

The following fact is a direct consequence of Fact 1.6.

Fact 1.10. *The number of permutation of $[n]$ is $n!$.*

Exercise 1.3

(1) Let $S(r,n)=\left\{{r \atop n}\right\}$. Then $\left\{{n \atop k}\right\}=\left\{{n-1 \atop k-1}\right\}+k\left\{{n-1 \atop k}\right\}$. (give a Combinatorial proof.)

(2) Let $s(n,k)=(-1)^{n-k}\left[{n \atop k}\right]$. Then $\left[{n \atop k}\right]=\left[{n-1 \atop k-1}\right]+(n-1)\left[{n-1 \atop k}\right]$

1.3 The Binomial Theorem

Define $[x^k]f$ to be the coefficient of the term x^k in the polynomial $f(x)$.

Fact 1.11. *For $j=1,2,...,n$, let $f_j(x)=\sum_{k\in I_j}x^k$ where I_j is a set of non-negative integers, and let $f(x)=\prod_{j=1}^n f_j(x)$. Then, $[x^k]f$ equals the number of solutions $(i_1,i_2,...,i_n)$ to $i_1+i_2+...+i_n=k$, where $i_j\in I_j$.*

Fact 1.12. *Let $f_1,...,f_n$ be polynomials and $f=f_1f_2...f_n$. Then,*

$$[x^k]f=\sum_{i_1+\cdots+i_n=k,i_j\geq 0}\left(\prod_{j=1}^n[x^{i_j}]f_j\right).$$

Theorem 1.13 (The Binomial Theorem)**.** *For any real x and any positive integer n, we have*

$$(1+x)^n=\sum_{i=0}^n\binom{n}{i}x^i.$$

Proof 1. Let $f=(1+x)^n$. By Fact 1.11 we have $[x^k]f$ equals the number of solutions $(i_1,i_2,...,i_n)$ to $i_1+i_2+...+i_n=k$ where $i_j\in\{0,1\}$, so $[x^k]f=\binom{n}{k}$. ∎

Proof 2. By induction on n. When $n = 1$, it is trivial. If the result holds for $n-1$, then $(1+x)^n = (1+x)(1+x)^{n-1} = (1+x)\sum_{i=0}^{n-1}\binom{n-1}{i}x^i = \sum_{i=1}^{n-1}(\binom{n-1}{i}+\binom{n-1}{i-1})x^i + 1 + x^n$. Since $\binom{n-1}{i}+\binom{n-1}{i-1} = \binom{n}{i}$ and $\binom{n}{0} = \binom{n}{n} = 1$, we have $(1+x)^n = \sum_{i=0}^{n}\binom{n}{i}x^i$. ∎

Fact 1.14. $\binom{2n}{n} = \sum_{i=0}^{n}\binom{n}{i}^2 = \sum_{i=0}^{n}\binom{n}{i}\binom{n}{n-i}$.

Proof 1. Since $(1+x)^{2n} = (1+x)^n(1+x)^n$, by Fact 1.12, we have $\binom{2n}{n} = [x^n](1+x)^{2n} = \sum_{i=0}^{n}([x^i](1+x)^n)([x^{n-i}](1+x)^n) = \sum_{i=0}^{n}\binom{n}{i}\binom{n}{n-i} = \sum_{i=0}^{n}\binom{n}{i}^2$. ∎

Proof 2. (It is easy to find a combinatorial proof.) ∎

Exercise 1.4 (**Vandermonde's Convolution Formula**)

$$\binom{n+m}{k} = \sum_{j=0}^{k}\binom{n}{j}\binom{m}{k-j}.$$

Fact 1.15. (1).

$$\sum_{\textit{all even } k}\binom{n}{k} = \sum_{\textit{all odd } k}\binom{n}{k} = 2^{n-1}.$$

(2).

$$\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}$$

Proof. (1). We see that $(1+x)^n = \sum_{i=0}^{n}\binom{n}{i}$. Taking $x = 1$ and $x = -1$, we have

$$\sum_{\text{all even } k}\binom{n}{k} = \sum_{\text{all odd } k}\binom{n}{k} = 2^{n-1}.$$

(2). Let $f(x) = (1+x)^n = \sum_{k=0}^{n} x^k$. Then $f'(x) = n(1+x)^{n-1} = \sum_{k=0}^{n} k\binom{n}{k}x^{k-1}$. Let $x = 1$, then we have $\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}$. ∎

Definition 1.16. *Let $k_j \geq 0$ be integers satisfying that $k_1 + k_2 + \cdots + k_m = n$. We define*

$$\binom{n}{k_1, k_2, ..., k_m} := \frac{n!}{k_1!k_2!...k_m!}.$$

The following theorem is a generalization of the binomial theorem.

Theorem 1.17 (Multinomial Theorem). *For any reals $x_1, ..., x_m$ and any positive integer $n \geq 1$, we have*

$$(x_1 + x_2 + \cdots + x_m)^n = \sum_{k_1+k_2+\cdots+k_m=n,\ k_j\geq 0}\binom{n}{k_1, k_2, ..., k_m}x_1^{k_1}x_2^{k_2}\cdots x_m^{k_m}.$$

Proof. Omit. ∎

Exercise 1.4. Suppose $\sum_{i=1}^{m} k_i = n$ with $k_i \geq 1$ for all $i \in m$. Then

$$\binom{n}{k_1, k_2, ..., k_m} = \binom{n-1}{k_1-1, k_2, ..., k_m} + \cdots + \binom{n-1}{k_1, k_2, ..., k_m-1}.$$

1.4 Estimating binomial coefficients

Theorem 1.18. *For any integer $n \geq 1$, we have*

$$e\left(\frac{n}{e}\right)^n \leq n! \leq en\left(\frac{n}{e}\right)^n \tag{1.1}$$

where $e = \lim_{n\to\infty}(1+\frac{1}{n})^n$ is the Euler number.

Proof. We have

$$\ln(n!) = \sum_{i=1}^{n} \ln i \leq \int_1^{n+1} \ln x \, dx = (x\ln x - x)\Big|_{x=1}^{x=n+1} = (n+1)\ln(n+1) - n.$$

Then it follows that

$$n! \leq \frac{(n+1)^{n+1}}{e^n}.$$

Reset $n = n-1$, we have

$$(n-1)! \leq \frac{n^n}{e^{n-1}} \iff n! \leq ne\left(\frac{n}{e}\right)^n.$$

Similarly we have

$$\ln(n!) \geq \int_1^n \ln x \, dx = (x\ln x - x)\Big|_1^n = n\ln n - (n-1),$$

which implies that

$$n! \geq \frac{n^n}{e^{n-1}} = e\left(\frac{n}{e}\right)^n,$$

as desired. ∎

Modifying the above proof, we can obtain the following improvement.

Exercise 1.5

$$n! \leq e\sqrt{n}\left(\frac{n}{e}\right)^n.$$

Definition 1.19. *Define $f \sim g$ for functions f and g, if $\lim_{n\to\infty}\frac{f(n)}{g(n)} = 1$.*

The following formula is well-known.

Theorem 1.20 (Stirling's formula.). $n! \sim \sqrt{2\pi n}(\frac{n}{e})^n$.

It is easy to show the following two facts.

Fact 1.21. *Let n be a fix integer. We can view $\binom{n}{k}$ as a function with $k \in \{0, 1, 2, ..., n\}$. It is increasing when $k \leq \lfloor\frac{n}{2}\rfloor$, and decreasing when $k > \lfloor\frac{n}{2}\rfloor$. Therefore, $\binom{n}{k}$ achievers its maximum at $k = \lfloor\frac{n}{2}\rfloor$ or $\lceil\frac{n}{2}\rceil$.*

Fact 1.22. $\frac{2^n}{n} \leq \binom{n}{\lfloor\frac{n}{2}\rfloor} \leq 2^n$

Exercise 1.6. For any even integer $n > 0$, we have

$$\frac{2^n}{\sqrt{2n}} \le \binom{n}{n/2} \le \frac{2^n}{\sqrt{n}}.$$

If we are allowed to use Stirling's formula, then we can get

$$\binom{n}{\frac{n}{2}} \sim \sqrt{\frac{2}{\pi}} \frac{2^n}{\sqrt{n}}.$$

Fact 1.23. $\binom{n}{k} = \frac{(n)_k}{k!} \le \frac{n^k}{k!}$.

Exercise 1.7. $1 + x \le e^x$ holds for any real x.

Theorem 1.24. *For any integers $1 \le k \le n$, we have $(\frac{n}{k})^k \le \binom{n}{k} \le (\frac{en}{k})^k$.*

Proof. Since $\frac{n-i}{k-i} \ge \frac{n}{k}$ for each $0 \le i \le k-1$, we have

$$\binom{n}{k} = \frac{n \cdot (n-1) \cdots (n-k+1)}{k \cdot (k-1) \cdots 1} = \left(\frac{n}{k}\right) \cdot \left(\frac{n-1}{k-1}\right) \cdots \left(\frac{n-k+1}{k}\right) \ge \left(\frac{n}{k}\right)^k$$

For the upper bound, since $k! \ge e(\frac{k}{e})^k > (\frac{k}{e})^k$, by Fact 1.23 we have

$$\binom{n}{k} \le \frac{n^k}{k!} \le \left(\frac{en}{k}\right)^k,$$

as desired. ∎

We can also prove the following strengthening.

Theorem 1.25. *For any integers $1 \le k \le n$,*

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{k} \le \left(\frac{en}{k}\right)^k.$$

Proof. By the binomial theorem, we have

$$\binom{n}{0} + \binom{n}{1}x + \cdots + \binom{n}{k}x^k \le (1+x)^n$$

for any $0 < x \le 1$. Then for any $0 < x \le 1$, it gives that

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{k} \le \frac{\binom{n}{0}}{x^k} + \frac{\binom{n}{1}}{x^{k-1}} + \cdots + \frac{\binom{n}{k}}{1} \le \frac{(1+x)^n}{x^k}.$$

Taking $x = \frac{k}{n} \in (0, 1]$, we have

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{k} \le \frac{(1+x)^n}{x^k} \le \frac{e^{xn}}{x^k} = \left(\frac{en}{k}\right)^k,$$

as desired. ∎