Combinatorics

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1 Lecture 10. Ramsey number

Definition 1.1. *For any $k \geq 2$ and integers $s_1, s_2, ..., s_k \geq 2$, the Ramsey number $R_k(s_1, s_2, ..., s_k)$ is the least integer N such that any k-edge-coloring of K_N has a clique K_{s_i} of color i, for some $i \in [k]$.*

In last lecture, we know

- $$R(s,t) \leq \binom{s+t-2}{s-1},$$
- $$R(s,t) \leq R(s-1,t) + R(s,t-1),$$

and we introduce an application of Ramsey theorem as following:

Theorem 1.2 (Schur's Theorem)**.** *For $k \geq 2$, there exists some integer $N = N(k)$ such that any coloring $\varphi : [N] \to [k]$ contains $x, y, z \in [N]$ satisfying that $\varphi(x) = \varphi(y) = \varphi(z)$ and $x + y = z$.*

Using this theorem, Schur proved that the restricted version of Fermat's last problem in $\mathbb{Z}_p$ for sufficiently large prime p.

Theorem 1.3 (Schur)**.** *For any integer $m \geq 1$, there is an integer $p(m)$ such that for any prime $p \geq p(m)$, $x^m + y^m = z^m \pmod p$ has a nontrivial solution in $\mathbb{Z}_p$.*

Proof. For prime p, consider the multiplicative group $\mathbb{Z}_p^* = \{1, 2, ..., p-1\}$. Let g be a generator of $\mathbb{Z}_p^*$. Then for $x \in \mathbb{Z}_p^*$, there exists exactly one pair of integers (i, j) such that $x = g^{im+j} \pmod p$ for some $0 \leq j \leq m-1$ and $0 \leq im+j \leq p-2$. Then we define a coloring $\varphi : \mathbb{Z}_p^* \to \{0, 1, ..., m-1\}$ by letting $\varphi(x) = j$.

By Schur's Theorem, choose $p(m) = N(m)$, and for any $p \geq p(m)$, the coloring φ gives $x, y, z \in \mathbb{Z}_p^*$ satisfying $\varphi(x) = \varphi(y) = \varphi(z)$ and $x + y = z$. Let $x = g^{i_1 m+j}$, $y = g^{i_2 m+j}$, $z = g^{i_3 m+j}$ $\pmod p$. Then $x + y = z$ implies that

$$g^{i_1 m+j} + g^{i_2 m+j} = g^{i_3 m+j} \pmod p, \tag{1.1}$$

thus

$$g^{i_1 m} + g^{i_2 m} = g^{i_3 m} \pmod p.$$

Let $\alpha = g^{i_1}$, $\beta = g^{i_2}$, $\gamma = g^{i_3}$. We have

$$\alpha^m + \beta^m = \gamma^m \pmod p.$$

∎

Remark: Schur's theorem holds in $\mathbb{Z}$, but we need to restrict the calculation in a multiplication cyclic group when deducing equation (1.1).

Definition 1.4. *A* <u>*probability space*</u> *is a pair* (Ω, P)*, where* Ω *is a finite set and* $P : 2^\Omega \to [0,1]$ *is a function assigning a number in the interval* $[0,1]$ *to every subset of* Ω *such that*

(i) $P(\emptyset) = 0$,

(ii) $P(\Omega) = 1$, *and*

(iii) $P(A \cup B) = P(A) + P(B)$ *for disjoint sets* $A, B \subset \Omega$.

We say

- Any subset A of Ω is called an <u>event</u>, and $P(A) = \sum_{\omega \in \Omega} P(\{\omega\})$.
- A <u>random variable</u> is a function $X : \Omega \to R$
- The <u>expectation</u> of a random variable X is:

$$E[X] := \sum_{\omega \in \Omega} P(\{\omega\}) \cdot X(\omega).$$

<u>**The linearity of expectations:**</u> for any two random variables X and Y on Ω, we have

$$E[X + Y] = E[X] + E[Y].$$

Now we discuss the following basic form of the probabilistic methods in Combinatorics:

(i) Imagine we need to find some combinatorial object satisfying certain property, call it a "good" property. We consider a big family for candidates and randomly pick one from this family, call it a random object. If the probability that the random object has "good" property is positive, then there must exist "good" objects.

(ii) To compute the probability of being "good", we often compute the probability of being "bad" and aim to show that this probability of being "bad" is strictly less than 1.

Theorem 1.5. *Let* n, s *satisfy* $\binom{n}{s} \cdot 2^{1-\binom{s}{2}} < 1$. *Then* $R(s,s) > n$.

Proof. We need to find a 2-edge-coloring of K_n such that it has no monochromatic clique K_s.

Let Φ be the family of all 2-edge-colorings of K_n. Let $c \in \Phi$ be chosen uniformly at random. Then c is a random 2-edge-coloring of K_n, where each edge of K_n is colored by red and blue, each with probability $\frac{1}{2}$, independent of each other edge.

Let B be the event that this random 2-edge-coloring has no monochromatic K_s. We want to prove $P(B) > 0$. Consider its complement event $A = \Omega \backslash B$ and its probability $P(A)$, where A is the event that c has a monochromatic K_s. For any $S \in \binom{[n]}{s}$, let A_S be the event that S forms a monochromatic K_s for c. So $A = \cup_{S \in \binom{[n]}{s}} A_S$, and $P(A_S) = 2^{1-\binom{s}{2}}$.

Thus

$$P(A) = P\left(\cup_{S \in \binom{[n]}{s}} A_S\right) \leq \sum_{S \in \binom{[n]}{s}} P(A_S) = \binom{n}{s} 2^{1-\binom{s}{2}} < 1,$$

This shows that $P(B) > 0$. ∎

Corollary 1.6. $R(s,s) \geq \frac{1}{e\sqrt{2}} s 2^{\frac{s}{2}}$.

Proof. Let $n = \frac{1}{e\sqrt{2}} s 2^{\frac{s}{2}} \left(\frac{e}{2}\right)^{1/s}$. Recall that $\binom{n}{s} < \frac{n^s}{s!}$ and $s! \geq e\left(\frac{s}{e}\right)^s$, thus we have that

$$\binom{n}{s} 2^{1-\binom{s}{2}} < \frac{n^s}{e\left(\frac{s}{e}\right)^s} 2^{1-\binom{s}{2}} = 1.$$

So by the above theorem, we get

$$R(s,s) > n = \frac{1}{e\sqrt{2}} s 2^{\frac{s}{2}} \left(\frac{e}{2}\right)^{1/s} \geq \frac{1}{e\sqrt{2}} s 2^{\frac{s}{2}}.$$

∎

Definition 1.7. *The random graph $G(n,p)$ for some real $p \in (0,1)$ is a graph with vertex set $\{1, 2, ..., n\}$, where each of potential $\binom{n}{2}$ edges appears with probability p, independent of other edges.*

In the proof of the previous theorem, in fact we consider $G(n, 1/2)$.
Let A be the property we are interested in. Let

$$\begin{aligned} P(A) &= P(G(n, \frac{1}{2}) \text{ satisfies the property } A) \\ &= \frac{\text{the number of graphs with vertex set } [n] \text{ satisfying the property } A}{2^{\binom{n}{2}}}. \end{aligned}$$

So $P(A)$ is a function of n, taking value in $[0,1]$.

Definition 1.8. *We say the random graph $G(n, \frac{1}{2})$ almost surely satisfies property A, if*

$$\lim_{n \to +\infty} P_r(A) = 1.$$

If $\lim_{n\to+\infty} P_r(A) = 0$, then $G(n, \frac{1}{2})$ almost surely does not satisfy the property A.

Theorem 1.9. *Random graph $G(n, \frac{1}{2})$ almost surely is not bipartite.*

Proof. Let A be the event that $G(n, \frac{1}{2})$ is bipartite. For any $U \subseteq [n]$, let A_U be the event that all edges of G are between U and $[n] \backslash U$. Then $A = \bigcup_{U \subseteq [n]} A_U$. We have

$$P(A_U) = \frac{\text{the number of graphs satisfying } A_U}{2^{\binom{n}{2}}} = \frac{2^{|U|(n-|U|)}}{2^{\binom{n}{2}}} \leq \frac{2^{\frac{n^2}{4}}}{2^{\frac{n(n-1)}{2}}} = 2^{-\frac{n^2}{4}+\frac{n}{2}}.$$

So by the union bound,

$$0 \leq P(A) = P(\bigcup_{U\subseteq[n]} A_U) \leq \sum_{U \subseteq [n]} P(A_U) \leq 2^n \cdot 2^{-\frac{n^2}{4}+\frac{n}{2}} = 2^{-\frac{n^2}{4}+\frac{3n}{2}}.$$

Thus $\lim_{n\to+\infty} P(A) = 0$. ∎