Combinatorics

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1 The Probabilistic Method

Definition 1.1. *Let $\mathcal{F}$ be a family of subsets of set Ω. We say $\mathcal{F}$ is a k-family if all its subsets have size k.*

Example 1.2. *A 2-family is just a graph.*

Definition 1.3. *We say $\mathcal{F}$ is 2-colorable if there exists a function $f : \Omega \to \{blue,red\}$ such that every subset A in $\mathcal{F}$ is not monochromatic (i.e., each A contains at least one blue vertex and at least one red vertex.)*

Definition 1.4. *For any $k \in Z^+$, let $m(k)$ be the minimum number of subsets in a k-family $\mathcal{F}$ which is not 2-colorable.*

Therefore, we see that $m(k) \leq t$ if and only if there exists a k-family $\mathcal{F}$ of t subsets which is not 2-colorable, and $m(k) > t$ if and only if any k-family of t subsets can be 2-colorable.

Fact 1.5. $m(2) = 3$. *Consider the graph K_3.*

Theorem 1.6. *For any k, we have $m(k) > 2^{k-1} - 1$, i.e., any k-family $\mathcal{F}$ of $2^{k-1} - 1$ subsets can be 2-colorable.*

Proof. Given a k-family $\mathcal{F}$ of $2^{k-1} - 1$ subsets, we aim to find a function $f : \Omega \to \{\text{blue,red}\}$ such that any subset A in $\mathcal{F}$ has a blue vertex and a red vertex. We call such f "good".

Now we consider a random function $\varphi : \Omega \to \{\text{blue,red}\}$, that is, each $x \in \Omega$ is colored by blue or red with probability $\frac{1}{2}$, independent of other choices.

Let S be the event that the random function φ is good. Let $T = S^c$ be the complement, i.e., there exists a subset A in $\mathcal{F}$ which is monochromatic under φ. For each $A \in \mathcal{F}$, let T_A be the event that the subset A is monochromatic under φ. So

$$T = \bigcup_{A \in \mathcal{F}} T_A.$$

It is easy to see that

$$P(T_A) = 2(\frac{1}{2})^k = 2^{1-k}.$$

So by the union bound,

$$P(T) = P(\bigcup_{A \in \mathcal{F}} T_A) \leq \sum_{A \in \mathcal{F}} P(T_A) = |\mathcal{F}| 2^{1-k} < 1.$$

Therefore, we have

$$P(\varphi \text{ is good}) = P(S) = 1 - P(T) > 0.$$

Since

$$P(\varphi \text{ is good}) = \frac{\text{number of good functions}}{\text{total number of functions}}.$$

We know that there exists at least one good function $f : \Omega \to \{\text{blue,red}\}$. ∎

Definition 1.7. *Given a probability space (Ω, P), we say events $A_1, A_2, ..., A_k$ are* independent *if for any $I \subset [n]$, we have $P(\bigcap_{i\in I} A_i) = \prod_{i\in I} P(A_i)$.*

Definition 1.8. *A* tournament *on n vertices is a directed graph obtained from the clique K_n by assigning a direction to each edge of K_n. For any arc $i \to j$, we say i is the head and j is the tail of the arc.*

Definition 1.9. *A tournament T satisfies the property S_k if for any subset A of size k, there exists a vertex $u \in V(T) \setminus A$ such that $u \to x$ for any $x \in A$.*

Question 1.10. *For any $k \in Z^+$, can we find a tournament satisfying the property S_k?*

Theorem 1.11. *For any $k \in Z^+$, if $\binom{n}{k}(1 - \frac{1}{2^k})^{n-k} < 1$, then there exists a tournament on n vertices satisfying the property S_k.*

Proof. We prove this by considering a random tournament T on $[n]$, that is, for any pair $\{i, j\}$, the arc $i \to j$ occurs with probability $\frac{1}{2}$, independent of other choices. Let B be the event that T does not satisfy the property S_k. For $A \in \binom{[n]}{k}$, let B_A be the event that for every vertex $x \in [n] \setminus A$ there exists some $u \in A$ with $u \to x$. So

$$B = \bigcup_{A\in\binom{[n]}{k}} B_A.$$

For $x \in [n]\setminus A$, let $B_{A,x}$ be the event that there exists some $u \in A$ with $u \to x$. So

$$B_A = \bigcap_{x\in[n]\setminus A} B_{A,x}.$$

It is easy to see that for any $x \in [n] \setminus A$

$$P(B_{A,x}) = 1 - \left(\frac{1}{2}\right)^k.$$

Note that only the arcs between x and A will effect the event $B_{A,x}$, and these arcs for distinct vertices x's are disjoint. This explains that all events $B_{A,x}$ for all $x \in [n] \setminus A$ are independent. So

$$P(B_A) = P(\bigcap_{x\notin A} B_{A,x}) = \prod_{x\notin A} P(B_{A,x}) = \left(1 - \left(\frac{1}{2}\right)^k\right)^{n-k}.$$

Therefore,

$$P(B) \le \sum_{A\in\binom{[n]}{k}} P(B_A) \le \binom{n}{k}\left(1 - \left(\frac{1}{2}\right)^k\right)^{n-k} < 1.$$

Thus, $P(B^c) > 0$, i.e., there exists a tournament on $[n]$ satisfying property S_k. ∎

Corollary 1.12. *For any $k \in Z^+$, there exists a minimal $f(k)$ such that there exists a tournament on $f(k)$ vertices satisfying the property S_k.*

Example 1.13. *We have $f(3) \leq 91$, as $\binom{91}{3}(\frac{7}{8})^{88} < 1$.*