Combinatorics

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2020 Fall, USTC

The Linearity of Expectation

- For any two variables X, Y, we have $E[X+Y] = E[X] + E[Y]$.
- $P(X \geq E[X]) > 0$.
- $P(X \leq E[X]) > 0$.

Definition 1.1. *A set A is sum-free, if for any $x, y \in A, x + y \notin A$, i.e., $x + y = z$ has no solutions in A.*

Example: Both $\{\lfloor \frac{n}{2} \rfloor + 1, \lfloor \frac{n}{2} \rfloor + 2, ..., n\}$ and $\{$all odd integers in $[n]\}$ are two sum-free sets in $[n]$ of size $\lceil \frac{n}{2} \rceil$.

Exercise 1.2. *Show that the maximum size of a sum-free subset A in $[n]$ is $\lceil \frac{n}{2} \rceil$.*

Theorem 1.3. *For any set A of non-zero integers, there exists a sum-free subset $B \subseteq A$ with $|B| \geq \frac{|A|}{3}$.*

Proof. We choose a prime p large enough such that $p > |a|$ for any $a \in A$. Consider $Z_p = \{0, 1, ..., p-1\}$ and $Z_p^* = \{1, 2, ..., p-1\}$. We note that there is a large sum-free subset under $Z_p($ mod $p)$:

$$S = \left\{ \lceil \frac{p}{3} \rceil + 1, \lceil \frac{p}{3} \rceil + 2, ..., \lceil \frac{2p}{3} \rceil \right\}.$$

Claim: For any $x \in Z_p^*$, $A_x = \{a \in A : ax \pmod p \in S\}$ is sum-free.

Proof. Suppose that there are $a, b, c \in A_x$ satisfying $a + b = c$. But we also have $ax \pmod p \in S$, $bx \pmod p \in S$, $cx \pmod p \in S$ and $ax \pmod p + bx \pmod p = cx \pmod p$ in Z_p. This is a contradiction to that S is sum-free in Z_p. ∎

Next, we want to find some $x \in Z_p^*$ such that $|A_x| \geq \frac{|A|}{3}$. We choose $x \in Z_p^*$ uniformly at random, and we compute, $E\big[|A_x|\big]$, the expectation of $|A_x|$.

Note that $|A_x| = \sum\limits_{a \in A} 1_{\{ax \pmod p \in S\}}$. So

$$E\big[|A_x|\big] = E\Big[\sum_{a \in A} 1_{\{ax \pmod p \in S\}}\Big] = \sum_{a \in A} E\big[1_{\{ax \pmod p \in S\}}\big] = \sum_{a \in A} P\big(ax \pmod p \in S\big).$$

We observe that for a fixed $a \in A$, $\{ax : x \in Z_p^*\} = Z_p^*$. So $P\big(ax \pmod p \in S\big) = \frac{|S|}{|Z_p^*|} \geq \frac{1}{3}$. And thus, $E\big[|A_x|\big] = \sum\limits_{a \in A} \frac{1}{3} = \frac{|A|}{3}$. Then, we know that there exists a choice of $x \in Z_p^*$ such that $|A_x| \geq E\big[|A_x|\big] \geq \frac{|A|}{3}$. ∎

Definition 1.4. *Given a graph G, a dominating set A in G is a subset of $V(G)$ such that any $u \in V(G)\backslash A$ has a neighbor in A.*

Theorem 1.5. *Let G be a graph on n vertices and with minimum degree $\delta > 1$. Then G contains a dominating set of at most $\dfrac{1+\ln(1+\delta)}{1+\delta}n$ vertices.*

Proof. Take $p \in (0,1)$, whose value will be determined later. We pick each vertex in $V(G)$ with probability p uniformly at random. Let A be the set of those chosen vertices. Let B be the set of vertices $b \in V(G) \setminus A$, which has no neighbors in A. Then we can see that

- $A \cup B$ is a dominating set in G.
- $b \in B$ if and only if $\big(\{b\} \cup N_G(b)\big) \cap A = \emptyset$.

That is, $b \in B$ if and only if b and all neighbors of b are not picked. So

$$P(b \in B) = (1-p)^{1+d_G(b)} \leq (1-p)^{1+\delta} \leq e^{-p(1+\delta)},$$

where the last inequality holds since $1+x \leq e^x$. Then, we have

$$E\big[|B|\big] = E\Big[\sum_{b\in V(G)} 1_{\{b\in B\}}\Big] = \sum_{b\in V(G)} P(b\in B) \leq n \cdot e^{-p(1+\delta)}.$$

We also have $E\big[|A|\big] = np$. Thus,

$$E\big[|A\cup B|\big] \leq E\big[|A| + |B|\big] = E\big[|A|\big] + E\big[|B|\big] \leq n\big(p + e^{-p(1+\delta)}\big).$$

By calculus, we see that when $p = \dfrac{\ln(1+\delta)}{1+\delta}$, $p+e^{-p(1+\delta)}$ is minimized with value $\dfrac{1+\ln(1+\delta)}{1+\delta}$. So we pick $p = \dfrac{\ln(1+\delta)}{1+\delta}$ to get $E\big[|A\cup B|\big] \leq \dfrac{1+\ln(1+\delta)}{1+\delta}n$. Therefore there exists a choice of $A \cup B$ such that $|A\cup B| \leq E\big[|A\cup B|\big] \leq \dfrac{1+\ln(1+\delta)}{1+\delta}n$, where $A\cup B$ is a dominating set of G. ∎

Definition 1.6. *Let $\alpha(G)$ be the maximum size of an independent set in G.*

Theorem 1.7. *For any graph G, $\alpha(G) \geqslant \sum_{v\in V(G)} \dfrac{1}{d(v)+1}$ where $d(v)$ denotes the degree of v in G.*

Proof. Let $V(G) = [n]$. For $i \in [n]$, let N_i be the neighborhood of i in G. Let S_n be the family of all permutations $\pi : [n] \to [n]$.

Given a permutation $\pi \in S_n$, we say a vertex $i \in [n]$ is $\pi-$good, if $\pi(i) < \pi(j)$ for any $j \in N_i$. Let M_π be the set of all $\pi-$good vertices.

Claim: For any $\pi \in S_n$, M_π is an independent set in G.

Proof. Suppose that there are two vertices $i, j \in M_\pi$ with $ij \in E(G)$. Let $\pi(i) < \pi(j)$. Then j is not $\pi-$good, a contradiction. ∎

We pick an $\pi \in S_n$ uniformly at random, and compute $E\big[|M_\pi|\big]$. Since $|M_\pi| = \sum_{i\in[n]} 1_{\{i\ is\ \pi-good\}}$, we have $E\big[|M_\pi|\big] = \sum_{i\in[n]} P(i\ is\ \pi - good) = \sum_{i\in[n]} \frac{1}{d(i)+1}$. Thus there exists a permutation $\pi \in S_n$ such that $|M_\pi| \geq \sum_{i\in[n]} \frac{1}{d(i)+1}$. Then by the definition of $\alpha(G)$ and our claim, we can get that $\alpha(G) \geq \sum_{v\in V(G)} \frac{1}{d(v)+1}$ which completes the proof. ∎