Combinatorics

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1 The Probabilistic Method

Theorem 1.1. *For any graph G, $\alpha(G) \geqslant \sum_{v\in V} \frac{1}{d(v)+1}$ where $d(v)$ denotes the degree of v in G.*

Corollary 1.2. *For any graph G with n vertices and m edges, we have $\alpha(G) \geq \frac{n^2}{2m+n}$.*

Proof. Exercise. ∎

Corollary 1.3. *For any graph G with n vertices and average degree d (i.e., $d = \frac{2m}{n}$), then $\alpha(G) \geq \frac{n}{1+d}$.*

Definition 1.4. *Turán graph $T_r(n)$ on r parts is an n-vertex graph G such that $V(G) = V_1 \cup V_2 \cup ... \cup V_r$ and $|V_1| \leq |V_2| \leq ... \leq |V_r| \leq |V_1| + 1$, where $ab \in E(G)$ if and only if $a \in V_i$ and $b \in V_j$ for some $i \neq j$.*

$T_r(n)$ is a balanced complete r-partite graph.

Theorem 1.5 (Turán's Theorem approximate form)**.** *If G is K_{r+1}-free, then $e(G) \leqslant \frac{r-1}{2r}n^2$.*

Theorem 1.6 (Turán's Theorem exact form)**.** *If an n-vertex graph G is K_{r+1}-free, then $e(G) \leqslant \mathrm{ex}(T_r(n)) \approx \frac{r-1}{2r}n^2$*

We give two proofs for the approximate version of Turán's Theorem.

First proof. Using Corollary 1.3 (Exercise). ∎

Second proof. We are given an n-vertex K_{r+1}-free graph G, where $V(G) = [n]$. Consider a function $p : [n] \to [0, 1]$ such that

$$\sum_{i\in[n]} p_i = 1. \tag{1.1}$$

We want to find the maximum of $f(p) = \sum_{ij\in E(G)} p_i p_j$ over all such functions $p : [n] \to [0, 1]$. Suppose p is the function obtaining the maximum $f(p)$, and subject to this, the number of vertices i with $p(i) \neq 0$ is minimized.
Claim. $\{i : p(i) > 0\}$ is a clique in G.

Proof. Suppose NOT, say $p(i), p(j) > 0$ and $ij \notin E(G)$. Let $S_i = \sum_{k\in N_G(i)} p_k$ and $S_j = \sum_{k\in N_G(j)} p_k$. Let $S_i \geqslant S_j$. Then we can assign a new function $p^* : [n] \to [0, 1]$ such that

$$p^*(i) = p(i) + p(j),\ p^*(j) = 0 \text{ and } p^*(k) = p(k) \text{ for } k \in [n] \setminus \{i, j\}.$$

Now we can compete

$$f(p^*) = f(p) - (p_iS_i + p_jS_j) + (p_i + p_j)S_i = f(p) + (S_i - S_j)p_j \geqslant f(p).$$

By the choice of p, we see $f(p^*) = f(p)$, but p^* has fewer vertices i with positive weight than p, a contradiction. This proves the claim. ∎

Let $1, 2, ..., s \in V(G)$ be vertices with positive weight. Then by the claim, we see $G[S] = K_s$, where $s \leq r$ as G is K_{r+1}-free. Then

$$\begin{aligned}\max_p f(p) =& \frac{1}{2}[(\sum_{1\leq i\leq s} p(i))^2 - \sum_{1\leq i\leq s} p^2(i)] = \frac{1}{2}[1 - \sum_{1\leq i\leq s} p^2(i)] \leq \frac{1}{2}\left[1 - s\left(\frac{\sum_{1\leq i\leq s} p(i)}{s}\right)^2\right] \\ =& \frac{1}{2}(1 - \frac{1}{s}) \leq \frac{1}{2}(1 - \frac{1}{r}).\end{aligned}$$

On the other hand,

$$\max_p f(p) \geq \frac{e(G)}{n^2}.$$

Combining, we have

$$e(G) \leq \frac{r-1}{2r} \cdot n^2.$$

∎

2 The Deleting Method

Previously, we often define an appropriate probability space and then show the random structure with desired property occurs with positive probability.

Today, we extend this idea and consider situation where random structure does not always have the desired property, and may have some very few "blemishes". The point that we want to make here is that after deleting all blemishes, we will obtain the wanted structure.

First we prove a half-way bound of Corollary 1.3.

Theorem 2.1. *Let G be a graph on n vertices and with average degree d. Then $\alpha(G) \geq \frac{n}{2d}$.*

Proof. Let $S \subset V(G)$ be a random subset, where for any $v \in V$, $P(v \in S) = p$. The value of p will be given later.

Let $X = |S|$ and $Y = e(S)$, Then $\mathbb{E}[X] = np$, $\mathbb{E}[Y] = mp^2$ where $m = \frac{nd}{2}$. So

$$\mathbb{E}[X - Y] = np - p^2 \cdot \frac{nd}{2} = n(p - \frac{d}{2}p^2).$$

By taking $p = \frac{1}{d}$, we have $\mathbb{E}[X - Y] = \frac{n}{2d}$. So there is a subset $S \subseteq V(G)$ such that $|S| - e(S) \geq \mathbb{E}[X - Y] = \frac{n}{2d}$. Now we delete one vertex for each edge of S. This leaves a subset $S^* \subseteq S$. Since all edges of S are destroyed, S^* must be an independent set of size at least $|S| - e(S) \geq \frac{n}{2d}$. ∎

Recall: If $\binom{n}{k}2^{1-\binom{k}{2}} < 1$, then Ramsey number $R(k, k) > n$. So $R(k, k) > \frac{1}{e\sqrt{2}}k2^{\frac{k}{2}}$.

Theorem 2.2. *For all n, $R(k, k) > n - \binom{n}{k}2^{1-\binom{k}{2}}$.*

Proof. Consider a random 2-edge-coloring of K_n, where each edge is colored by red or blue with probability $\frac{1}{2}$, independent of other choices. For $A \in \binom{[n]}{k}$, let X_A be the indicator random variable of the event that A induces a monochromatic K_k.

Let $X = \sum_{A \in \binom{[n]}{k}} X_A$ be the number of monochromatic k-subsets. Then we have

$$\mathbb{E}[X] = \sum_{A \in \binom{[n]}{k}} \mathbb{E}[X_A] = \binom{n}{k} 2^{1-\binom{k}{2}}.$$

So there exists a 2-edge-coloring of K_n, where the number of monochromatic k-subsets is at most $\mathbb{E}[X] = \binom{n}{k} 2^{1-\binom{k}{2}}$. Next we remove one vertex from each monochromatic k-subset. This will delete at most $X \leq \binom{n}{k} 2^{1-\binom{k}{2}}$ vertices and destroy all monochromatic k-subsets. So it remains at least $n - \binom{n}{k} 2^{1-\binom{k}{2}}$ vertices, which contains NO monochromatic K_k. ∎

Corollary 2.3.

$$R(k,k) > \frac{1}{e}(1+o(1))k 2^{\frac{k}{2}}.$$

Proof. Exercise, by maximizing $n - \binom{n}{k} 2^{1-\binom{k}{2}}$ for a fixed k. ∎

3 Markov's Inequality

Theorem 3.1 (Markov's Inequality)**.** *Let $X \geq 0$ be a random variable and $t > 0$, then $P(X \geq t) \leq \frac{\mathbb{E}[X]}{t}$.*

Corollary 3.2. *Let $X_n \geq 0$ be integer value random variable for $n \in \mathbb{N}^+$ in (Ω_n, P_n). If $\mathbb{E}[X_n] \to 0$ as $n \to +\infty$, then $P(X_n = 0) \to 1$ (as $n \to +\infty$), i.e., $X_n = 0$ almost surely occurs.*

Theorem 3.3. *For a random graph $G(n,p)$ where $p \in (0,1)$, then*

$$P\left(\alpha(x) \leq \lceil \frac{2 \ln n}{p} \rceil \right) \to 1 \quad as \quad n \to +\infty.$$

Proof. Let $k = \lceil \frac{2 \ln n}{p} \rceil$. For any $S \in \binom{[n]}{k+1}$, let A_S be the event that S is an independent set, and let X_S be the indicator random variable of the event A_S. Let $X_n = \sum_{S \in \binom{[n]}{n+1}} X_S$ be the number of independent set of size $k+1$. Then $P(\alpha(G) \leq k) = P(X_n = 0)$. Now we compute $\mathbb{E}[X_n]$ as following:

$$\begin{aligned}
\mathbb{E}[X_n] &= \sum_{S \in \binom{[n]}{k+1}} \mathbb{E}[X_S] = \binom{n}{k+1} (1-p)^{\binom{k+1}{2}} \\
&\leq \frac{n^{k+1}}{(k+1)!} e^{-p\binom{k+1}{2}} \\
&= \frac{1}{(k+1)!} (n e^{-p \cdot \frac{k}{2}})^{k+1} \\
&\leq \frac{1}{(k+1)!} \to 0.
\end{aligned}$$

By the corollary, we see that $P(\alpha(G) \le k) = P(X_n = 0) \to 1$ as $n \to +\infty$.

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Definition 3.4. *For a graph G, the* chromatic number $\chi(G)$ *is the minimum integer k such that $V(G)$ can be partitioned into k independent sets.*

Fact 3.5. *(1). $\chi(K_n) = n$,*
(2). $\chi(G) \le 2$ if and only if G is bipartite,
(3). $\chi(C_{2n+1}) = 3$.

Proposition 3.6. *For any graph G on n vertices, $\chi(G) \cdot \alpha(G) \ge n$.*

Definition 3.7. *The girth $g(G)$ of a graph G is the length of a shortest cycle in G.*

Theorem 3.8 (Erdős)**.** *For any $k \in \mathbb{N}^+$, there exists a graph G with $\chi(G) \ge k$ and $g(G) \ge k$.*

Proof. Consider a random graph $G = G(n,p)$ where p will be determined later. Let $t = \lceil \frac{2 \ln n}{p} \rceil$, by the previous theorem, $\alpha(G) \le t$ almost surely occurs.

Let X_n be the number of cycles of length less than k in G. Then

$$\mathbb{E}[X_n] = \sum_{i=3}^{k-1} \frac{n(n-1)\cdots(n-i+1)}{2i} \cdot p^i,$$

where $\frac{n(n-1)\cdots(n-i+1)}{2i}$ is the number of $C_i's$ in K_n. So

$$\mathbb{E}[X_n] \le \sum_{i=3}^{k-1} (np)^i = \frac{(np)^k - 1}{np - 1}.$$

By Markov's inequality,

$$P(X_n > \frac{n}{2}) \le \frac{E[X_n]}{n/2} \le \frac{2[(np)^k - 1]}{n(np-1)}.$$

Let $p = n^{-\frac{k-1}{k}}$. So $np = n^{\frac{1}{k}}$. Then

$$P(X_n > \frac{n}{2}) \le \frac{2(n-1)}{n(n^{\frac{1}{k}} - 1)} \to 0 \quad as \quad n \to +\infty.$$

So there exists a graph G on n vertices such that $X_n \le n/2$ and $\alpha(G) \le t = \lceil \frac{2 \ln n}{p} \rceil \le 3 \ln n \cdot n^{\frac{k-1}{k}}$.

By deleting one vertex from each cycle of length at most $k-1$, we can find an induced subgraph G^* of G, which has at least $\frac{n}{2}$ vertices and NO cycles of length at most $k-1$. Moreover,

$$\alpha(G^*) \le \alpha(G) \le 3 \ln n \cdot n^{\frac{k-1}{k}}.$$

By proposition 3.6, we have

$$\chi(G^*) \ge \frac{|V(G^*)|}{\alpha(G^*)} \ge \frac{n/2}{3(\ln n) n^{\frac{k-1}{k}}} \ge \frac{n^{1/k}}{6 \ln n} \ge k \text{ and } g(G^*) \ge k.$$

∎