Combinatorics

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1 Odd/Even town

Question. A town has n residents. They want to form some clubs according to the following rules:

(i) Each club has an odd number of members.

(ii) Every 2 clubs must share an even number of members.

How many clubs can they form?

Examples. (a) $A_i = \{i\}$ for $i \in [n] \Rightarrow n$ clubs.

(b) n is even, $A_i = [n] \setminus \{i\} \Rightarrow n$ clubs.

(c) n is even, $A_1 = [n] \setminus \{1\}$, $A_2 = [n] \setminus \{2\}$, $A_i = \{1, 2, i\}$ for $i \in \{3, ..., n\} \Rightarrow n$ clubs.

Theorem 1.1 (Odd/Even town)**.** *Let $\mathcal{F} \subseteq 2^{[n]}$ be a family satisfying:*

(i) $|A|$ is odd for all $A \in \mathcal{F}$,

(ii) $|A \cap B|$ is even, for all $A \neq B \in \mathcal{F}$.

Then $|\mathcal{F}| \leq n$.

Proof. For each $A \in \mathcal{F}$, we define an indicator vector $\vec{\mathbb{1}}_A \in \mathbb{F}_2^n = \{0,1\}^n$ such that

$$\vec{\mathbb{1}}_A(i) = \begin{cases} 1, & \text{if } i \in A \\ 0, & \text{if } i \notin A, \end{cases}$$

where $\mathbb{F}_2$ is the finite field of size 2. Then, these conditions become

$$\begin{cases} \vec{\mathbb{1}}_A \cdot \vec{\mathbb{1}}_A = 1, & \forall\, A \in \mathcal{F} \\ \vec{\mathbb{1}}_A \cdot \vec{\mathbb{1}}_B = 0, & \forall\, A \neq B \in \mathcal{F}. \end{cases}$$

Next, we claim that these vectors $\vec{\mathbb{1}}_A$ in $\mathbb{F}_2^n$ are linearly independent.
Let $\alpha_A \in \mathbb{F}_2$, such that $\sum_{A \in \mathcal{F}} \alpha_A \vec{\mathbb{1}}_A = \vec{0}$. Then for any $B \in \mathcal{F}$,

$$0 = \vec{0} \cdot \vec{\mathbb{1}}_B = (\sum_{A \in \mathcal{F}} \alpha_A \vec{\mathbb{1}}_A) \cdot \vec{\mathbb{1}}_B = \sum_{A \in \mathcal{F}} \alpha_A (\vec{\mathbb{1}}_A \cdot \vec{\mathbb{1}}_B) = \alpha_B \cdot \vec{\mathbb{1}}_B \cdot \vec{\mathbb{1}}_B = \alpha_B.$$

This proves the claim. Therefore the number of vectors $\vec{\mathbb{1}}_A$'s is at most the dimension of $\mathbb{F}_2^n$, which is n. So $|\mathcal{F}| \leq n$. ∎

2 Even/Odd town

Theorem 2.1 (Even/Odd town)**.** *Let $\mathcal{F} \subseteq 2^n$ be such that:*

(i) $|A|$ is even, for all $A \in \mathcal{F}$,

(ii) $|A \cap B|$ is odd, for all $A \neq B \in \mathcal{F}$.

Then $|\mathcal{F}| \leq n$.

First we show a weaker result:

Lemma 2.2. *Such $\mathcal{F}$ satisfies $|\mathcal{F}| \leq n+1$.*

Proof. Adding a new element $n+1$ to each set $A \in \mathcal{F}$ to get a new family $\mathcal{F}^*$. We see $\mathcal{F}^*$ satisfies the Odd/Even town conditions. So $|\mathcal{F}| = |\mathcal{F}^*| \leq n+1$. ∎

Now we give the proof of Theorem 2.1.

Proof of Theorem 2.1. It suffices to prove that $|\mathcal{F}| \neq n+1$. Suppose for a contradiction that $\mathcal{F} = \{A_1, A_2, \cdots, A_{n+1}\}$. For each $A_i \in \mathcal{F}$, define $\vec{\mathbb{1}}_{A_i} \in \mathbb{F}_2^n$ as before. So we have $n+1$ vectors in an n-dimension space. Thus, they must be linearly dependent. Therefore, there exist $\alpha_i \in \mathbb{F}_2$ for $1 \leq i \leq n+1$ which are not all 0's such that

$$\sum_{i=1}^{n+1} \alpha_i \vec{\mathbb{1}}_{A_i} = \vec{0}.$$

We also have

$$\begin{cases} \vec{\mathbb{1}}_A \cdot \vec{\mathbb{1}}_A = 0, & \forall\, A \in \mathcal{F} \\ \vec{\mathbb{1}}_A \cdot \vec{\mathbb{1}}_B = 1, & \forall\, A \neq B \in \mathcal{F}. \end{cases}$$

Then for each $1 \leq j \leq n+1$,

$$0 = \vec{0} \cdot \vec{\mathbb{1}}_{A_j} = \left(\sum_{i=1}^{n+1} \alpha_i \vec{\mathbb{1}}_{A_i} \right) \cdot \vec{\mathbb{1}}_{A_j} = \sum_{i=1}^{n+1} \alpha_i - \alpha_j.$$

So $\alpha_j = \sum_{i=1}^{n+1} \alpha_i$ for all $1 \leq j \leq n+1$. They are all equal. Because all α_j's can not be all 0's, we derive that $\alpha_j = 1$ for all $1 \leq j \leq n+1$ and n must be even. Moreover,

$$\sum_{i=1}^{n+1} \vec{\mathbb{1}}_{A_i} = \vec{0}. \tag{1}$$

Consider $\mathcal{F}^c = \{A^c : A \in \mathcal{F}\}$, we will see that $\mathcal{F}^c$ also satisfies the Even/Odd town conditions:

- $|A^c| = n - |A|$ is even, for all $A \in \mathcal{F}$.
- $|A^c \cap B^c| = n - |A \cup B| = n - |A| - |B| + |A \cap B|$ is odd, for all $A \neq B \in \mathcal{F}$.

By the same proof, we can derive that

$$\sum_{i=1}^{n+1} \vec{\mathbb{1}}_{A_i^c} = \vec{0}. \tag{2}$$

Now (1)+(2) gives that

$$\vec{0} = \sum_{i=1}^{n+1} (\vec{\mathbb{1}}_{A_i} + \vec{\mathbb{1}}_{A_i^c}) = (n+1)\vec{1} = \vec{1},$$

a contradiction. ∎

Exercise 2.3 (Even/Even-town)**.** Let $\mathcal{F} \subset 2^{[n]}$ be such that:

(i) $|A|$ =even, for all $A \in \mathcal{F}$,

(ii) $|A \cap B|$ =even, for all $A \neq B \in \mathcal{F}$.

Then $|\mathcal{F}| \leq 2^{n/2}$. (let n be even)

3 Fisher's Inequality

Theorem 3.1 (Fisher's Inequality)**.** *For a fixed k, let $\mathcal{F} \subseteq 2^{[n]}$ be a family such that $|A \cap B| = k$, for all $A \neq B \in \mathcal{F}$. Then, $|\mathcal{F}| \leq n$.*

Proof. For each $A \in \mathcal{F}$, define vector $\vec{\mathbb{1}}_A \in R^n$ as before. Then for any $A, B \in \mathcal{F}$, $\vec{\mathbb{1}}_A \cdot \vec{\mathbb{1}}_B = k$. Again, we want to show $\vec{\mathbb{1}}_A$'s are linearly independent over $\mathbb{R}^n$. Let $\sum_{A \in \mathcal{F}} \alpha_A \vec{\mathbb{1}}_A = \vec{0}$, where $\alpha_A \in \mathbb{R}$. Then

$$0 = \left(\sum_{A \in \mathcal{F}} \alpha_A \vec{\mathbb{1}}_A\right) \cdot \left(\sum_{A \in \mathcal{F}} \alpha_A \vec{\mathbb{1}}_A\right) = \sum_{A \in \mathcal{F}} \alpha_A^2 \vec{\mathbb{1}}_A \cdot \vec{\mathbb{1}}_A + \sum_{A \neq B} \alpha_A \alpha_B \vec{\mathbb{1}}_A \cdot \vec{\mathbb{1}}_B$$

$$= \sum_{A \in \mathcal{F}} \alpha_A^2 |A| + k \cdot \sum_{A \neq B} \alpha_A \alpha_B = k \left(\sum_{A \in \mathcal{F}} \alpha_A\right)^2 + \sum_{A \in \mathcal{F}} \alpha_A^2 (|A| - k) \geq 0,$$

where the last inequality holds because each A is of size at least k. This implies that $\sum_{A \in \mathcal{F}} \alpha_A = 0$ and $\alpha_A^2(|A| - k) = 0$ for all $A \in \mathcal{F}$. Since $|A \cap B| = k$ for any $A \neq B \in \mathcal{F}$, we have at most one set A of size exactly k. Call this subset A^* if exists. Thus for each $A \in \mathcal{F} \setminus \{A^*\}$, $\alpha_A = 0$. However $\sum_{A \in \mathcal{F}} \alpha_A = 0$, we derive that all $\alpha_A = 0$. Thus all $\vec{\mathbb{1}}_A$'s are independent and then $|\mathcal{F}| \leq n$. ∎

Lemma 3.2. *Suppose P is a set of n points in $\mathbb{R}^2$. Then either they are in a line, or they define at least n lines.*

Proof. Let L be the family of all lines defined by P. We want to show that $|L| = 1$ or $|L| \geq n$ For each point $x_i \in P$, define $L_i = \{\ell \in L : \text{the line } \ell \text{ passes through } x_i\}$. Note that for all $i \neq j$, $|L_i \cap L_j| = 1$. We also observe that there exist $i \neq j$ with $L_i = L_j$ if and only if all n points lie in a line. Therefore, either $|L| = 1$, or for any $x_i, x_j \in P$, we have $L_i \neq L_j$. We may assume that the second case occurs. Let $\mathcal{F} = \{L_i : x_i \in P\}$. Clearly, $\mathcal{F}$ satisfies the conditions of Fisher's inequality, so we can derive that $n = |\mathcal{F}| \leq |L|$. ∎

Lemma 3.3. *Let G be a graph whose vertices are triples in $\binom{[k]}{3}$ such that for any two $A, B \in \binom{[k]}{3}$, $A \sim_G B$ iff $|A \cap B| = 1$. Then G doesn't contain any clique or independent set of size k+1.*

Proof. Consider the maximum clique of G, say using vertices $A_1, A_2, ..., A_m \in \binom{[k]}{3}$ with $|A_i \cap A_j| = 1$, for $1 \leq i < j \leq m$. By Fisher's inequality, $m \leq k$.

Now consider the maximum independent set of G, say consisting of vertices $B_1, B_2, ..., B_t \in \binom{[k]}{3}$. We see $|B_i| = 3$ is odd and $|B_i \cap B_j| = 0$ or 2 is even. By Odd/Even-town, we have $t \leq k$. ∎

Corollary 3.4. $R(k+1, k+1) > \binom{k}{3}$.

Remark. This gives us an explicit construction for Ramsey number $R(k+1, k+1)$.

Note that this bound is much weaker than previous bound $R(k+1, k+1) > c \cdot k2^{\frac{k}{2}}$.

4 1-Distance Problem

Problem 1 (1-Distance Problem)**.** Given n points in $\mathbb{R}^2$, what is the maximum number of pairs of distance 1?

Theorem 4.1. *There are at most $O(n^{\frac{3}{2}})$ pairs at distance 1.*

Proof. Define a graph G on n points as following: for points a, b, $a \sim b$ iff $d(a, b) = 1$.

We claim that G is $K_{2,3}$-free. Since the neighbors of the point a must lie on the circle with center a and with radius 1, and any such 2 circles can intersect at most 2 points, then they show that G is $K_{2,3}$-free.

Thus the number of pairs at distance 1 is

$$e(G) \leq \text{ex}(n, K_{2,3}) = O(n^{\frac{3}{2}}).$$

∎

Exercise 4.2.

$$\text{ex}(n, K_{2,3}) = O(n^{\frac{3}{2}}).$$

Open problem (Erdős)**.** Can one find an example of n points in $\mathbb{R}^2$ with n^{1+c} pairs at distance 1 for $c > 0$?

Problem 2. What is the maximum number of points in $\mathbb{R}^n$ such that the distance between any two points is 1?

Theorem 4.3. *There are at most $n+1$ points in $\mathbb{R}^n$ such that the distance between any two points is 1.*

Proof. Assume we have $m+1$ such points in $\mathbb{R}^n$. We assume one of them is $\vec{0}$ and let others be $\vec{v_1}, \vec{v_2}, ..., \vec{v_m} \in \mathbb{R}^n$. Then we have

- $\vec{v_i} \cdot \vec{v_i} = ||\vec{v_i} - \vec{0}||^2 = 1$ for $i \in [m]$,
- $\vec{v_i} \cdot \vec{v_j} = \frac{1}{2}$, for any $i \neq j \in [m]$,

because $1 = ||\vec{v_i} - \vec{v_j}||^2 = ||\vec{v_i}||^2 + ||\vec{v_j}||^2 - 2\vec{v_i} \cdot \vec{v_j} = 1 + 1 - 2\vec{v_i} \cdot \vec{v_j}$.

Consider the matrix

$$A = \begin{pmatrix} \vec{v}_1 \\ \vec{v}_2 \\ \vdots \\ \vec{v}_m \end{pmatrix}_{m \times n}.$$

So

$$A \cdot A^T = \begin{pmatrix} 1 & \frac{1}{2} & \cdots & \frac{1}{2} \\ \frac{1}{2} & 1 & \cdots & \frac{1}{2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2} & \frac{1}{2} & \cdots & 1 \end{pmatrix}_{m \times m}.$$

Since $\det(A \cdot A^T) \neq 0$, we get $\text{rank}(A \cdot A^T) = m$. Then $n \geq \text{rank } A \geq \text{rank}(A \cdot A^T) = m$. So $m \leq n$ as desired. ∎

Remark: we can also apply this method for the Even/Odd town.

Definition 4.4. A 2-distance set is a set of points in $\mathbb{R}^n$ whose pairwise distance is either c or d for some $c, d > 0$.

Problem (2-Distance Problem)**.** What is the maximum size of a 2-distance set?

Instead of considering vectors, we also can define polynomials of certain degree.

Lemma 4.5. *Let $f_i : \Omega \to \mathbb{F}$ be polynomials for $i \in [n]$, where $\mathbb{F}$ is a field. If there are $v_i \in \Omega$ for $i \in [n]$ such that*

$$\begin{cases} f_i(v_i) \neq 0, & \forall\, i \in [n] \\ f_i(v_j) = 0, & \forall\, j < i, \end{cases}$$

then $f_1, f_2, ..., f_n$ are linear independent over $\mathbb{F}^\Omega$.

Proof. Exercise. ∎

Theorem 4.6. *Any 2-distance set in $\mathbb{R}^n$ has at most $\frac{1}{2}(n+1)(n+4)$ points.*

(To be continued.)