Combinatorics

Instructor: Jie Ma, Scribed by Jun Gao, Jialin He and Tianchi Yang

2020 Fall, USTC

1 The Algebraic Method

Definition 1.1. *A 2-distance set is a set of points in* $\mathbb{R}^n$ *whose pairwise distance is either* c *or* d *for some* $c, d > 0$.

In the previous approach, we define a vector $\vec{1}_A$ for each $A \in \mathcal{F}$. Instead of considering vectors, one also can define certain polynomials, as polynomials of certain degree also form a vector space.

Lemma 1.2. *For* $i \in [n]$*, let* $f_i : \Omega \to \mathbb{F}$ *be polynomial, where* $\mathbb{F}$ *is a field. If there are elements* $v_i \in \Omega$ *for* $i \in [n]$ *satisfying*

$$\begin{cases} f_i(v_i) \neq 0, & \forall i \\ f_i(v_j) = 0, & \forall j < i, \end{cases}$$

then $f_1, f_2, ..., f_n$ *are linear independent over the "linear space" spanned by polynomials* $f : \Omega \to \mathbb{F}$.

Theorem 1.3. *Any 2-distance set in* $\mathbb{R}^n$ *has at most* $\frac{1}{2}(n+1)(n+4)$ *points.*

Proof. Let $A = \{\vec{a}_1, \vec{a}_2, ..., \vec{a}_m\}$ be such a set with distances $c > 0$, $d > 0$. For each $i \in [m]$, define $f_i(\vec{x}) = (||\vec{x} - \vec{a}_i||^2 - c^2)(||\vec{x} - \vec{a}_i||^2 - d^2)$ for $\vec{x} \in \mathbb{R}^n$. Then

$$\begin{cases} f_i(\vec{a}_i) = c^2 d^2 \neq 0, & \forall i \\ f_i(\vec{a}_j) = (||\vec{a}_j - \vec{a}_i||^2 - c^2)(||\vec{a}_j - \vec{a}_i||^2 - d^2) = 0, & \forall j \neq i. \end{cases}$$

By Lemma 1.2, $f_1, f_2, ..., f_m$ are linearly independent in the "linear space" that contains $f_1, ..., f_m$. We want to bound the dimension of "some vector space" which contains all polynomials $f_1, f_2, ..., f_m$.

Let $\vec{x} = (x_1, x_2, ..., x_n), \vec{a}_i = (a_{i1}, ..., a_{in})$. Note that

$$\begin{aligned} f_j(\vec{x}) &= (\sum_i (x_i - a_{ji})^2 - c^2)(\sum_i (x_i - a_{ji})^2 - d^2) \\ &= (\sum_i x_i^2 - 2\sum_i x_i a_{ji} + \sum_i a_{ji}^2 - c^2)(\sum_i x_i^2 - 2\sum_i x_i a_{ji} + \sum_i a_{ji}^2 - d^2), \end{aligned}$$

can be expressed as the linear combination of the following polynomials:

$$B = \{(\sum_i x_i^2)^2, x_j(\sum_i x_i^2), x_i x_j, x_i, 1\}.$$

We see that B contains $1 + n + \binom{n}{2} + n + n + 1 = \frac{n(n-1)}{2} + 3n + 2 = \frac{(n+1)(n+4)}{2}$ elements and each f_i is contained in the linear space spanned by B. So $|A| = m$ is at most the dimension of *span*(B), which is at most $\frac{(n+1)(n+4)}{2}$.

∎

Remark 1.4. *This proof can be extended to k-distance Problem.*

Next, we consider a generalization of Fisher's inequality.

Definition 1.5. *Consider a subset $L \subseteq \{0, 1, 2, ..., n\}$. We say a family $\mathcal{F} \subseteq 2^{[n]}$ is L-intersecting, if for any $A \neq B \in \mathcal{F}$, $|A \cap B| \in L$.*

Theorem 1.6 (Frankl-Wilson, 1981)**.** *If $\mathcal{F} \subseteq 2^{[n]}$ is an L-intersecting family, then $|\mathcal{F}| \leq \sum_{k=0}^{|L|} \binom{n}{k}$.*

Proof. Let $\mathcal{F} = \{A_1, A_2, ..., A_m\}$ where $|A_1| \leq |A_2| \leq \cdots \leq |A_m|$. For each $i \in [m]$, define $f_i(\vec{x}) : \mathbb{R}^n \to \mathbb{R}^n$ by

$$f_i(\vec{x}) = \prod_{\ell \in L, \ell < |A_i|} (\vec{x} \cdot \vec{1}_{A_i} - \ell).$$

Consider the indicator vectors $\vec{1}_{A_1}, \vec{1}_{A_2}, ..., \vec{1}_{A_m}$. Then we have

- $f_i(\vec{1}_{A_i}) = \prod_{\ell \in L, \ell < |A_i|}(|A_i| - \ell) > 0,$
- $f_i(\vec{1}_{A_j}) = \prod_{\ell \in L, \ell < |A_i|}(|A_i \cap A_j| - \ell) = 0.$

This is because we have $\ell = |A_j \cap A_i| \in L$ and $\ell < |A_i|$ for some ℓ (as $j < i$, $|A_j| \leq |A_i|$). By Lemma 1.2, we see that $f_1, f_2, ..., f_m$ are linear independent.

Next we want to define some new polynomials $\tilde{f}_i(\vec{x})$ from f_i such that $\tilde{f}_1, \tilde{f}_2, ..., \tilde{f}_m$ are remain linearly independent, but these $\tilde{f}_i(\vec{x})$'s lie in a "better" linear space.

Observer that all vector $\vec{1_{A_j}}$ are 0/1-vectors. Let $\tilde{f}_i(\vec{x})$ be a new polynomial obtained from $f_i(\vec{x})$ by replacing all terms x_j^k (for $k \geq 1$) by x_j.

For any 0/1-vectors $\vec{y}$, we have $\tilde{f}_i(\vec{y}) = f_i(\vec{y})$. This shows that $\tilde{f}_1, \tilde{f}_2, ..., \tilde{f}_m$ are also linearly independent. And we see each $\tilde{f}_i(\vec{x})$ is a linear combination of the monomials $\prod_{i \in I} x_i$ for $I \in [n]$ with $|I| \leqslant |L|$ (as deg$\tilde{f}_i \leq$ deg$f_i \leq |L|$). Clearly the number of such monomials is at most $\sum_{k=0}^{|L|} \binom{n}{k}$ which is also the dimension of the space containing $\tilde{f}_1, \tilde{f}_2, ..., \tilde{f}_m$. This prove that

$$|\mathcal{F}| = |m| \leq \sum_{k=0}^{|L|} \binom{n}{k}.$$

∎

Theorem 1.7. *Let p be a prime and $L \subseteq \mathbb{F}_p = \{0, 1, ..., p-1\}$. Let $\mathcal{F} \subseteq 2^{[n]}$ be a family satisfying that*

- $|A| \notin L \pmod p$ *fro any* $A \in \mathcal{F}$,
- $|A \cap B| \in L \pmod p$ *for all* $A \neq B \in \mathcal{F}$.

Then $|\mathcal{F}| \leq \sum_{k=0}^{|L|} \binom{n}{k}$.

Proof. All operations are mod p. Let $\mathcal{F} = \{A_1, ..., A_m\}$. Define $f_i(\vec{x}) : \mathbb{F}_p^* \to \mathbb{F}_p$ be such that

$$f_i(\vec{x}) = \prod_{\ell \in L} (\vec{x} \cdot \vec{1}_{A_i} - \ell).$$

Then

- $f_i(\vec{1}_{A_i}) = \prod_{\ell\in L}(|A_i| - \ell) \neq 0$,
- $f_i(\vec{1}_{A_j}) = \prod_{\ell\in L}(|A_i \cap A_j| - \ell) = 0$ for all $i \neq j$.

So $f_1, f_2, ..., f_m$ are linearly independent over Z_p^n. Then repeating the proof of Theorem 1.6, we get the desired bound.

∎

Now we prove an application of these results.

Theorem 1.8 (Frankl-Wilson)**.** *For any prime p, there is a graph G on $n = \binom{p^3}{p^2-1}$ vertices such that both of the maximum clique and the maximum independent set are at most $\sum\limits_{i=0}^{p-1} \cdot \binom{p^3}{i}$*

Proof. Let $G = (V, E)$ be the following graph, where $V = \binom{[p^3]}{p^2-1}$, and for $A, B \in V$, $A \sim_G B$ if and only if $|A \cap B| \not\equiv p-1 \pmod p$.

Consider the maximum clique with vertices set $A_1, A_2, ..., A_m \in \binom{[p^3]}{p^2-1}$. Thus we have

- $|A_i \cap A_j| \not\equiv p-1 \pmod p$, for $i \neq j$,
- $|A_i| = p^2 - 1 \equiv p-1 \pmod p$.

By Theorem 1.7 with $L = \{0, 1, 2, ..., p-2\} \subseteq \mathbb{F}_p$ we can derive that $m \leqslant \sum\limits_{i=0}^{p-1} \binom{p^3}{i}$.

Consider the maximum independent set $B_1, B_2, ..., B_t$. Then we have $|B_i \cap B_j| = p-1 \pmod p$ for all $i \neq j$, implying that $|B_i \cap B_j| \in \{p-1, 2p-1, ..., p(p-1)-1\} = L^*$ with $|L^*| = p-1$. Thus $B_1, B_2, ..., B_t$ is L^*-intersecting family in $\binom{[p^3]}{p^2-1}$. By Theorem 1.6, we have $t \leqslant \sum_{i=0}^{p-1} \binom{p^3}{i}$. ∎

Corollary 1.9.

$$R(k+1, k+1) \geq k^{\Omega(\log(k)/\log(\log(k))}.$$

Proof. Use the construction from Theorem 1.8. Let $k = \sum_{i=0}^{p-1} \binom{p^3}{i}$. So $R(k+1, k+1) > n$. We have that

$$k = \sum_{i=0}^{p-1} \binom{p^3}{i} \simeq \binom{p^3}{p} \simeq (p^2)^p \simeq p^{2p},\ n \simeq (\frac{p^3}{p^2})^{p^2} \simeq p^{p^2},$$

which implies that

$$\log(k) \simeq p \log(p),\ \ \log(\log(k)) \simeq \log(p),$$

so

$$p \simeq \frac{\log(k)}{\log(\log(k))}.$$

Then we have

$$n = \binom{p^3}{p^2-1} \simeq (p^{2p})^{p/2} \simeq k^p = k^{\Omega(\log(k)/\log(\log(k)))}.$$

∎

Definition 1.10. *Given a set $S \subseteq R^n$, the* diameter *of S is defined as $Diam(S) = \sup\{d(x, y) : x, y \in S\}$ where $d(x, y)$ denotes the Euclidean distance between x and y in R^n.*

If $Diam(S) < +\infty$, then we say S is bounded.
Borswk's Conjecture: Every bounded $S \subseteq R^d$ can be partitioned into $d+1$ sets of strictly smaller diameter.

Remark 1.11. *This was verified for all $S \subseteq R^d$ with $d \leqslant 3$ and for the S is a sphere and any $d \geq 2$.*

Lemma 1.12. *For any prime p, there is a set $\mathcal{F}$ of $\frac{1}{2}\binom{4p}{2p}$ vectors in $\{-1,1\}^{4p}$ such that every subset of size $2\binom{4p}{p-1}$ vectors contains an orthogonal pair of vectors.*

Proof. Let Q=$\{I \in \binom{[4p]}{2p} : 1 \in I\}$, then $|Q| = \frac{1}{2}\binom{4p}{2p}$. For any $I \in Q$, define $\vec{v}^I \in \{-1,1\}^{4p}$ by

$$\vec{v}_i = \begin{cases} 1, & i \in I \\ -1, & i \notin I. \end{cases}$$

Let $\mathcal{F} = \{\vec{v}^I : I \in Q\}$ with $|\mathcal{F}| = |Q| = \frac{1}{2}\binom{4p}{2p}$.
Claim 1. $\vec{v}^I \perp \vec{v}^J$ if and only if $|I \cap J| \equiv 0 \pmod{p}$.

Proof. $\vec{v}^I \perp \vec{v}^J$ if and only if $\vec{v}^I \cdot \vec{v}^J = 0$. Since $\vec{v}^I \cdot \vec{v}^J = |I \cap J| - |I^C \cap J| - |I \cap J^C| + |I^C \cap J^C|$ (we have $|I \cap J| = |I^C \cap J^C|$ as $|I| = |J| = 2p$), we have that $\vec{v}^I \perp \vec{v}^J$ if and only if $|I \Delta J| = 2p = 4p - 2|I \cap J|$ if and only if $|I \cap J| = p$

Since $1 \in I \cap J$ and $|I| = |J| = 2p$, we have $\vec{v}^I \perp \vec{v}^J$ if and only if $|I \cap J| \equiv 0 \pmod{p}$. ∎

Claim 2. For any subset $\mathcal{F}' \subseteq \mathcal{F}$ without orthogonal pairs, $|\mathcal{F}'| \leq \sum_{k=0}^{p-1} \binom{4p}{k} < 2\binom{4p}{p-1}$.

Proof. Let $Q' = \{I \in Q : \vec{v}^I \in \mathcal{F}'$. By Claim 1, Q' is a subfamily of $\binom{[4p]}{2p}$ satisfying

- $|A| = 2p \equiv 0 \pmod{p}, \forall A \in Q'$,
- $|A \cap B| \neq 0 \pmod{p}, \forall A \neq B \in Q'$.

By Theorem 1.8 (with $L = \{1, 2, ..., p-1\}$), we get $|\mathcal{F}'| = |Q'| \leq \sum_{k=0}^{p-1} \binom{4p}{k}$. ∎

Now the conclusion of Lemma follows by Claim 2. ∎

Definition 1.13. *The* tensor product *of a vectors $\vec{v} \in \mathbb{R}^n$ is $\vec{w} = \vec{v} \otimes \vec{v} \in \mathbb{R}^{n^2}$ by $w_{ij} = v_i \cdot v_j$ for all $1 \leq i, j \leq n$.*

Theorem 1.14 (Kahn-Kalai, 1993)**.** *For sufficiently large d, there exists a bounded set $S \subset \mathbb{R}^d$(a finite set) such that any partition of S into $1.1^{\sqrt{d}}$ subsets contains a subset of the same diameter.*

Proof. Take the family $\mathcal{F}$ from the above lemma. So $\mathcal{F} \subset \{-1,1\}^n \subset \mathbb{R}^n$ (with $n = 4p$). Let $X = \{\vec{v} \otimes \vec{v} : \vec{v} \in \mathcal{F}\} \subseteq \mathbb{R}^{n^2}$. Let $d = n^2 = (4p)^2 = 16p^2$. For any $\vec{w} = \vec{v} \otimes \vec{v} \in X$,

$$||\vec{w}||^2 = \sum_{1 \leq i,j \leq n} w_{ij}^2 = \sum_{1 \leq i,j \leq n} v_i^2 v_j^2 = (\sum_{i=1}^n v_i^2)(\sum_{j=1}^n v_j^2) = n^2,$$

and thus $||\vec{w}|| = n$.

For $\vec{w} = \vec{v} \otimes \vec{v}, \vec{w}' = \vec{v}' \otimes \vec{v}' \in X$, we have

$$\vec{w} \cdot \vec{w}' = \sum_{1 \le i,j \le n} w_{ij} w'_{ij} = \sum_{1 \le i,j \le n} (v_i v'_i)(v_j v'_j) = (\sum v_i v'_i)^2 = (\vec{v} \cdot \vec{v}')^2.$$

This says that $\vec{w} \perp \vec{w}'$ if and only if $\vec{v} \perp \vec{v}'$. Thus,

$$||\vec{w} - \vec{w}'||^2 = ||\vec{w}||^2 + ||\vec{w}'||^2 - 2\vec{w} \cdot \vec{w}' = 2n^2 - 2(\vec{v} \cdot \vec{v}')^2 \le 2n^2,$$

this proves that $Diam(X) = \sqrt{2}n$ and $|X| = |\mathcal{F}| = \frac{1}{2}\binom{[4p]}{2p}$.

By Lemma 1.12, any subset of $2\binom{4p}{p-1}$ vectors in $\mathcal{F}$ contains an orthogonal pair of vector $\vec{v}, \vec{v}'$. Thus, any subset of $2\binom{4p}{p-1}$ vectors in X must contain a pair $\vec{w} = \vec{v} \otimes \vec{v}, \vec{w}' = \vec{v}' \otimes \vec{v}'$ with $\vec{v} \perp \vec{v}'$, which give the maximum distance $||\vec{w} - \vec{w}'|| = \sqrt{2}n$. Thus to decrease the diameter, we must partition X into subsets of size less than $2\binom{4p}{p-1}$, so the number of subsets needed is at least

$$\frac{|X|}{2\binom{4p}{p-1}} = \frac{\frac{1}{2}\binom{4p}{2p}}{2\binom{4p}{p-1}} = \frac{1}{4}\frac{(3p+1)\cdots(2p+1)}{(2p)\cdots(p)} \ge \frac{1}{4} \cdot (\frac{3}{2})^{p+1} \ge C \cdot (\frac{3}{2})^{\frac{\sqrt{d}}{4}} \ge 1.1^{\sqrt{d}},$$

where $d = n^2 = 16p^2$ is the dimension of X. ∎