Combinatorics

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2020 Fall, USTC

1 Bollobás' Theorem

We first recall the following theorem which we learned in week 5.

Sperner's Theorem: Let $\mathscr{F} \subseteq 2^{[n]}$ be a family such that for any $A \neq B \in \mathscr{F}$, $A \not\subseteq B$, and $B \not\subseteq A$, then $|\mathscr{F}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$.

LYM-inequality: For such $\mathscr{F}$, $\sum\limits_{A \in \mathscr{F}} \frac{1}{\binom{n}{|A|}} \leq 1$.

Theorem 1.1 (Bollobás' Theorem)**.** *Let $A_1, A_2, ..., A_m$ and $B_1, B_2, ..., B_m$ be the subsets of some ground set Ω. If we have*

(1) $A_i \cap B_j \neq \emptyset$, for any $i \neq j \in [m]$,

(2) $A_i \cap B_i = \emptyset$, for any $i \in [m]$.

Then $\sum\limits_{i=1}^{m} \frac{1}{\binom{a_i+b_i}{a_i}} \leq 1$, where $a_i = |A_i|$, and $b_j = |B_j|$.

Remark 1.2. *The condition (1): $A_i \cap B_j \neq \emptyset$, for any $i \neq j$ cannot be weakened to $i < j$; otherwise we have the following counterexamples:*

- *$m = 2$, $A_1 = B_2 = \{1\}$ and $A_2 = B_1 = \emptyset$.*

  We can see that $\sum\limits_{i=1}^{m} \frac{1}{\binom{a_i+b_i}{a_i}} = 2 > 1$.

- *$m = 3$, $A_1 = B_2 = \{1\}$, $A_2 = A_3 = B_1 = \{3\}$, and $B_3 = \{1, 2\}$.*

  We can see that $\sum\limits_{i=1}^{m} \frac{1}{\binom{a_i+b_i}{a_i}} = \frac{4}{3} > 1$.

Proposition 1.3. *Bollobás' Theorem can imply LYM-inequality and LYM-inequality will imply Sperner's Theorem.*

Proof. We first show that Bollobás' Theorem can imply the LYM-inequality. Let $\mathscr{F} \subseteq 2^{[n]}$ satisfy that $A \not\subseteq B$, and $B \not\subseteq A$ for any $A \neq B \in \mathscr{F}$. Let $\mathscr{F} = \{A_1, A_2, ..., A_m\}$ and $\mathscr{F}' = \{B_1, B_2, ..., B_m\}$, where $B_i = [n] \backslash A_i$. We now varify that $A_1, ..., A_m$ and $B_1, ..., B_m$ satisfy the conditions (1) and (2).

- $A_i \cap B_j = A_i \backslash A_j \neq \emptyset$, for any $i \neq j \in [m]$,
- $A_i \cap B_i = \emptyset$, for any $i \in [m]$.

So by Bollobás' Theorem:

$$1 \geq \sum_{i=1}^{m} \frac{1}{\binom{a_i+b_i}{a_i}} = \sum_{A \in \mathscr{F}} \frac{1}{\binom{n}{|A_i|}}.$$

Note that LYM-inequality can easily imply the Sperner's Theorem and we are done. ∎

Proof of Bollobás' Theorem. Let $X = \bigcup_{i=1}^{m}(A_i \cup B_i)$ and let $n = |X|$. We will prove by induction on n. Base case: $n = 1$ ($A_1 = \{1\}, B_1 = \emptyset$) is clear.

Now we assume this statement holds for $|X| \leq n-1$. Let $I_x = \{i \in [m] : x \notin A_i\}$ for any $x \in X$. Define $\mathscr{F}_x = \{A_i : i \in I_x\} \cup \{B_i \backslash \{x\} : i \in I_x\}$. Since each set in $\mathscr{F}_x$ doesn't contain x, we see that $|\cup_{S \in \mathscr{F}_x} S| \leq |X \setminus \{x\}| \leq n-1$. Moreover, the family $\mathscr{F}_x$ satisfy the induction hypothesis. Hence by induction, we get

$$\sum_{i \in I_x} \frac{1}{\binom{|A_i|+|B_i \backslash \{x\}|}{|A_i|}} \leq 1, \text{ for any } x \in X.$$

We sum up the above inequalities for all $x \in X$ and get

$$\sum_{x \in X} \sum_{i \in I_x} \frac{1}{\binom{|A_i|+|B_i \backslash \{x\}|}{|A_i|}} \leq n. \tag{1.1}$$

For each $i \in [m]$, it contributes either 0, or $\frac{1}{\binom{a_i+b_i}{a_i}}$ or $\frac{1}{\binom{a_i+b_i-1}{a_i}}$ to each x. The term $\frac{1}{\binom{a_i+b_i}{a_i}}$ occurs when $i \in I_x$ and $x \notin B_i$, i.e., $x \notin A_i \cup B_i$ which occur exactly $(n - a_i - b_i)$ times. The term $\frac{1}{\binom{a_i+b_i-1}{a_i}}$ occurs when $i \in I_x$ and $x \in B_i$, i.e., $x \in B_i$ which occur exactly b_i times. Therefore, we see that (1.1) is equivalent to

$$\sum_{i=1}^{m} \left((n - a_i - b_i) \frac{1}{\binom{a_i+b_i}{a_i}} + b_i \frac{1}{\binom{a_i+b_i-1}{a_i}} \right) \leq n.$$

Since we have $\frac{1}{\binom{a_i+b_i-1}{a_i}} = \frac{1}{\binom{a_i+b_i}{a_i}} \cdot \frac{a_i+b_i}{b_i}$, which implies that

$$n \sum_{i=1}^{m} \frac{1}{\binom{a_i+b_i}{a_i}} \leq n,$$

as claimed. ∎

Definition 1.4. *Let $\mathbb{F}$ be a field. A set $A \subseteq \mathbb{F}^n$ is in general position, if any n vectors in A are linearly independent over $\mathbb{F}$.*

Example 1.5. *For $a \in \mathbb{F}$, let $\vec{m}(a) = (1, a, a^2, ..., a^{n-1}) \in \mathbb{F}^n$ be a moment curve. Then $\{\vec{m}(a) : a \in \mathbb{F}\}$ is in general position (because of the Vandermonde matrix).*

Next, we use the so-called "general position" argument to prove the skew version of Bollobás' Theorem, where the condition (1) is relaxed to $i < j$.

Theorem 1.6. *(The skew version of Bollobás' Theorem) Let $A_1, ..., A_m$ be the sets of size r and $B_1, ..., B_m$ be the sets of size s such that*

- *$A_i \cap B_j \neq \emptyset$, for any $i < j$,*
- *$A_i \cap B_i = \emptyset$, for any $i \in [m]$.*

Then $m \leq \binom{r+s}{s}$.

Proof. Let X=$\bigcup_{i\in[m]}(A_i \cup B_i)$. Take a set V of vectors $\vec{v} = (v_0, v_1, ..., v_r)$ in $\mathbb{R}^{r+1}$ such that V is in general position and $|V| = |X|$. Then we identify the elements of X with vectors of V. From now on, we may view each A_i or B_j as a subset in $V \subseteq \mathbb{R}^{r+1}$, where $|A_i| = r$ and $|B_j| = s$. For each $j \in [m]$, we define $f_j(\vec{x}) = \prod_{\vec{v}\in B_j} \vec{x} \cdot \vec{v}$ for any $\vec{x} \in \mathbb{R}^{r+1}$. So

$$f_j(\vec{x}) = \prod_{\substack{\vec{v}=(v_0,...,v_r) \\ \vec{v}\in B_j}} (v_0x_0 + \cdots + v_rx_r),$$

where $\vec{x} = (x_0, ..., x_r) \in \mathbb{R}^{r+1}$. Note that $f_j(\vec{x})$ is generated by the following monomials $x_0^{i_0}x_1^{i_1}\cdots x_r^{i_r}$, where $i_0 + i_1 + \cdots + i_r = s$ and $i_j \geq 0$ for $0 \leq j \leq r$. There are exactly $\binom{s+r}{r}$ such monomials, so $f_1, f_2, .., f_m$ are contained in a polynomial linear space of dimension $\binom{s+r}{r}$. It suffices to prove that $f_1, f_2, .., f_m$ are linearly independent. Note that

$$f_j(\vec{x}) = 0 \text{ if and only if there exists some } \vec{v} \in B_j \text{ such that } \vec{v} \cdot \vec{x} = 0. \tag{1.2}$$

Consider the linear subspace Span(A_i), which is spanned by the r vectors in A_i. Since $A_i \subseteq V \subseteq \mathbb{R}^{r+1}$ and V is in general position, we see that all r vectors in A_i are linearly independent and thus $dim\big(Span(A_i)\big) = r$. So $(Span(A_i))^\perp$ has dimension 1. We choose $\vec{a}_i \in (Span(A_i))^\perp$ for each $i \in [m]$. Then for each $\vec{v} \in V$,

$$\vec{v} \cdot \vec{a}_i = 0 \;\text{ if and only if }\; \vec{v} \in Span(A_i) \;\text{ if and only if }\; \vec{v} \in A_i. \tag{1.3}$$

Because, otherwise the $r + 1$ vectors in $\{\vec{v}\} \cup A_i$ are linearly dependent, contradicting that V is in general position.

Combining (1.2) and (1.3), $f_j(\vec{a}_i) = \prod_{\vec{v}\in B_j} \vec{v} \cdot \vec{a}_i = 0$ if and only if there exists $\vec{v} \in B_j$ such that $\vec{v} \cdot \vec{a}_i = 0$ which is equivalent to say that there exists $\vec{v} \in B_j \cap A_i$, i.e., $A_i \cap B_j \neq \emptyset$. Thus we get the following

$$\begin{cases} f_j(\vec{a}_i) = 0, \text{ for any } i < j, \\ f_i(\vec{a}_i) \neq 0, \text{ for any } i. \end{cases}$$

By the previous lemma, we now see that $f_1, ..., f_m$ are linearly independent. ∎

2 Covering by complete bipartite subgraphs

Problem. Determine the minimum $m = m(n)$ such that the edge set $E(K_n)$ of a clique K_n can be partitioned into a disjoint union of edge sets of m complete subgraphs of K_n.

Fact 2.1. $m(n) \leq n - 1$.

Proof. Because we can express $E(K_n)$ as a disjoint union of $n - 1$ stars. ∎

We remark that there are more than one way to partition $E(K_n)$ into $n-1$ complete bipartite subgraphs.

Theorem 2.2 (Graham-Pollak)**.** $m(n) = n - 1$.

Proof. Suppose that $E(K_n) = E(B_1) \cup E(B_2) \cup \cdots \cup E(B_m)$, where $B_1, B_2, ..., B_m$ are complete bipartite subgraphs on $[n]$. We want to show that $m \geq n - 1$. Let X_i and Y_i be the two parts of B_i. For B_k, we define an $n \times n$ matrix $A_k = (a_{ij}^{(k)})_{n \times n}$ by

$$a_{ij}^{(k)} = \begin{cases} 1, & \text{if } i \in X_k \text{ and } j \in Y_k, \\ 0, & \text{otherwise.} \end{cases}$$

It is clear to see that $rank(A_k) = 1$ for any k. Let $A = \sum_{k=1}^{m} A_k$, implying $rank(A) \leq \sum_{k=1}^{m} rank(A_k) = m$. Then $A + A^T = J_n - I_n$, where $J_n = (1)_{n \times n}$, because each $ij \in E(K_n)$ belongs to exactly one of the graphs B_k, where we have $a_{ij}^{(k)} = 0$ and $a_{ji}^{(k)} = 1$ or $a_{ij}^{(k)} = 1$ and $a_{ji}^{(k)} = 0$. It suffices to show that $rankA \geq n - 1$.

Suppose for a contradiction that $rankA \leq n - 2$. Let A' be the $(n+1) \times n$ matrix obtained from A by adding an extra row $(11 \cdots 1)$, so $rank(A') \leq n - 1$. Then there exists a non-zero vector $\vec{x} \in \mathbb{R}^n$ such that $A'\vec{x} = \vec{0} \in \mathbb{R}^{n+1}$, which is equivalent to $A\vec{x} = \vec{0} \in \mathbb{R}^n$ and $\vec{1} \cdot \vec{x} = 0$, where $\vec{x} = (x_1, ..., x_n)$. Consider $\vec{x}^T(A + A^T)\vec{x} = \vec{x}^T(J_n - I_n)\vec{x}$ implying that $0 = \vec{x}^T J_n \vec{x} - \vec{x}^T \vec{x} = 0 - \sum_{i=1}^{n} x_i^2 < 0$, a contradiction. This proves that $n - 1 \leq rankA \leq m$. ∎

3 Finite Projective Plane (FPP)

Definition 3.1. *Let X be a finite set and $\mathscr{L} \subseteq 2^X$ be a family. The pair $(X, \mathscr{L})$ is called a finite projective plane (FPP for short) if it satisfies the following three properties.*

(P0) There exists a 4-set $F \subseteq X$ such that $|F \cap L| \leq 2$ for any $L \in \mathscr{L}$.

(P1) Any two $L_1, L_2 \in \mathscr{L}$ has $|L_1 \cap L_2| = 1$.

(P2) For any two $x_1, x_2 \in X$, there exists exactly one subset $L \in \mathscr{L}$ with $\{x_1, x_2\} \subseteq L$.

We call the elements of X as points, and the sets in $\mathscr{L}$ as lines. Let us explain the three properties:

- (P0) is used to exclude some non-interesting cases.
- (P1) says that any two lines intersect at exactly one point.
- (P2) says that any two points determine one line.

Example 3.2 (The Fano plane (the smallest FPP))**.** *Where the set $X = [7]$ has 7 points and the set $\mathscr{L}$ has 7 lines with $\mathscr{L} = \{\{1,2,3\}, \{3,4,5\}, \{1,5,6\}, \{1,4,7\}, \{2,5,7\}, \{3,6,7\}, \{2,4,6\}\}$.*

Proposition 3.3. *Let $(X, \mathscr{L})$ be an FPP. Then any two lines $L, L' \in \mathscr{L}$ satisfy $|L| = |L'|$.*

Proof. We claim that there exists a point $x \in X$ with $x \notin L \cup L'$. To see this, let $F \subseteq X$ be from $(P0)$. Then $|F \cap L| \leq 2$, $|F \cap L'| \leq 2$. So we may assume that $F = \{a, b, c, d\}$ and $F \cap L = \{a, b\}$, $F \cap L' = \{c, d\}$. Let $\overline{ac}$ denote the line in $\mathscr{L}$ containing a and c; similarly, define $\overline{bd}$. Let $z \in \overline{ac} \cap \overline{bd}$ be the unique point. If $z \notin L \cup L'$, then we are done. So we may assume $z \in L$, i.e., $z \in L \cap \overline{ac}$. But $a \in L \cap \overline{ac}$, which implies that $z = a$. But again, we see $a, b \in L \cap \overline{bd}$, a contradiction.

For any point $\ell \in L$, the line $\overline{x\ell}$ intersects with L' at the unique point, say $\ell' \in L'$. We define a mapping $\phi : L \to L'$ by letting $\phi(\ell) = \ell'$ for any $\ell \in L$. Next we show that ϕ is a bijection between L and L'. (Exercise) ∎

Definition 3.4. *Let $(X, \mathscr{L})$ be a finite projective plane. The order of $(X, \mathscr{L})$ is the number $|L| - 1$, for each $L \in \mathscr{L}$.*

Proposition 3.5. *Let $(X, \mathscr{L})$ be a FPP of order n. Then*

(1) For each $x \in X$, there are exactly $n+1$ lines passing through x.

(2) $|X| = n^2 + n + 1$.

(3) $|\mathscr{L}| = n^2 + n + 1$.

Proof. (1). Consider $x \in X$. Let F be the 4-set satisfying $(P0)$. Let $a, b, c \in F \backslash \{x\}$. Then, at least one of the lines $\overline{ab}, \overline{ac}$ which doesn't contains x (otherwise, a, b, c, x are in the same line). Let L be such a line with $x \notin L$. Let $L = \{x_0, x_1, ..., x_n\}$. Then $\overline{x_i x}$ define $n+1$ lines. On the other hand, any line passing through x must intersect at some point say x_i. Thus, there are exactly $n+1$ lines containing x.

(2). By (1), there are $n+1$ lines $L_0, L_1, ..., L_n$ containing x. It is clear that $(L_i \backslash \{x\}) \cap (L_j \backslash \{x\}) = \emptyset$ for any $i \neq j$. Thus, $|L_0 \cup L_1 \cup \cdots \cup L_n| = n(n+1) + 1 = n^2 + n + 1$. It is easy to see that $X = L_0 \cup L_1 \cup \cdots \cup L_n$.

(3). Let the incidence graph of a FPP $(X, \mathscr{L})$ be the bipartite graph with two parts X and L, where $x \in X$ is adjacent to $L \in \mathscr{L}$ if and only if $x \in L$. This defines an $(n+1)$-regular bipartite graph. So $|\mathscr{L}| = |X| = n^2 + n + 1$. ∎