Combinatorics

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1 Finite Projective Plane (FPP)

Definition 1.1. *The* incidence graph *of a FPP $(X, \mathcal{L})$ is a bipartite graph G with parts X and $\mathcal{L}$, where $x \in X$ and $L \in \mathcal{L}$ are adjacent in G if and only if $x \in L$.*

Definition 1.2. *The* dual *$(\mathcal{L}, \wedge)$ of a FFP $(X, \mathcal{L})$ is obtained by taking the incidence graph G of $(X, \mathcal{L})$ and interpreting the points in $(X, \mathcal{L})$ as the lines in the new FPP and the lines in $(X, \mathcal{L})$ as the points in the new FPP.*

Remark 1.3. *For any $x \in X$, let $L_x = \{L \in \mathcal{L} : x \in L\}$ be a new line in $(\mathcal{L}, \wedge)$. So $\wedge = \{L_x : x \in X\}$.*

Proposition 1.4. *The dual $(\mathcal{L}, \wedge)$ of any FPP $(X, \mathcal{L})$ of order n is also a FPP of order n.*

Proof. We point out that $(P1)$ for $(X, \mathcal{L})$ gives rise to $(P2)^*$ for $(\mathcal{L}, \wedge)$ and $(P2)$ for $(X, \mathcal{L})$ gives rise to $(P1)^*$ for $(\mathcal{L}, \wedge)$.

$(P1)$: for any $L_1, L_2 \in \mathcal{L}$ satisfying $L_1 \cap L_2 = \{x\}$ for some $x \in X$.

$(P2)$: For any two points $x_1, x_2 \in X$ there exists exactly one subset $L \in \mathcal{L}$ with $\{x1, x2\} \subseteq Ł$.

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$(P2)^*$: for any $L_1, L_2 \in \mathcal{L}$ satisfying $L_1 \cap L_2 = \{x\}$ for some $x \in X$.

We consider $(P0)^*$ for $(\mathcal{L}, \wedge)$.

$(P0)^*$: there exist four new points in $(\mathcal{L}, \wedge)$ such that any three of them cannot be contained in a new line of $(\mathcal{L}, \wedge)$, i.e., there exist $L_1, L_2, L_3, L_4 \in \mathcal{L}$ such that no L_x contains any three of them if and only if there exist $L_1, L_2, L_3, L_4 \in \mathcal{L}$ such that No three of them contains a point $x \in X$.

Consider the 4-set $F = \{a, b, c, d\} \in (X, \mathcal{L})$ satisfying $(P0)^*$. Note that $|F \cap \mathcal{L} \leq 2|$ for any $L \in \mathcal{L}$, So we have four distinct lines $L_1 = \overline{ab}, L_2 = \overline{cd}, L_3 = \overline{ac}, L_4 = \overline{bd}$.

It is easy to check that these four lines satisfy $(P0)^*$. ∎

Theorem 1.5. *A finite projective plane of order n exists whenever a field with n elements exists.*

And we know that a field with n elements exists if and only if $n = p^k$ for a prime p.

Open Conjecture. A FPP of order n exists if and only if n is a power of a prime.

We know this holds for $n \leq 11$. In particular, FPP of $n = 10$ does not exists. It is open for $n = 12$.

Next we introduce an application of FPP in Turán numbers. Recall the following result.

Theorem 1.6. *Any m-vertex C_4-free graph G has $e(G) \leq \frac{m}{4}(1 + \sqrt{4m - 3})$.*

Theorem 1.7. *For infinitely many integers m, there exists a C_4-free graph on m vertices with at least $0.35m^{3/2}$.*

Proof. Take any FPP $(X, \mathcal{L})$ of order n, and consider its incidence graph G. Note that G has $m = 2(n^2 + n + 1)$ vertices and

$$e(G) = (n^2 + n + 1)(n + 1) \geq (n^2 + n + 1)^{\frac{3}{2}} = (\frac{m}{2})^{\frac{3}{2}} \geq 0.35m^{3/2}.$$

It is clear that G is C_4-free by the property of FPP. ∎