Combinatorics

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1 Integer partitions

How many ways are there to write a natural number n as a sum of several natural numbers? The total number of *ordered partitions* of n is $\sum_{1\leq k\leq n}\binom{n-1}{k-1}=2^{n-1}$. Here "ordered partition" means that we will view $1+1+2, 1+2+1$ as two different partitions of 4.

We then consider the *unordered partitions*. For instance, we will view $1+2+3$ and $3+2+1$ of 6 as the same one.

Let p_n be the number of unordered partitions of n. So $p_1=1$, $p_2=2$, $p_3=3$ and $p_4=5$. We have the following theorem.

Theorem 1.1. *The generating function $P(x)$ of $\{p_n\}_{n\geq 0}$ is an infinite product of polynomials*

$$P(x)=\prod_{k=1}^{+\infty}\frac{1}{1-x^k}.$$

Proof. Let n_j be the number of the j's in such a partition of n. Then it holds that

$$\sum_{j\geq 1} j\cdot n_j=n.$$

If we use i_j to express the contribution of the addends equal to j in a partition of n (i.e., $i_j=j\cdot n_j$), then

$$\sum_{j\geq 1} i_j=n, \quad \text{where } \ i_j\in\{0,j,2j,3j,...\}.$$

Note that in the above summation, j can run from 1 to infinity, or run from 1 to n. So by the fact we discussed earlier, p_n is the coefficient of x^n in the product

$$P(x)=(1+x+x^2+...)(1+x^2+x^4+...)...(1+x^n+x^{2n}+...)...=\prod_{k=1}^{+\infty}\frac{1}{1-x^k}.$$

This finishes the proof of this theorem. ∎

2 The Catalan number

First let us recall the definition of $\binom{r}{k}$ for real numbers r and positive integers k, and the Newton's binomial Theorem. We obtained that

$$\binom{\frac{1}{2}}{k}=\frac{(-1)^{k-1}2}{4^k}\cdot\frac{(2k-2)!}{k!(k-1)!}.$$

Let n-gon be a polygon with n corners, labelled as corner 1, corner 2,..., corner n.

Definition 2.1. *A triangulation of the n-gon is a way to add lines between corners to make triangles such that these lines do not cross inside of the polygon.*

Then we have the following theorem.

Theorem 2.2. *The total number of triangulations of the $(k+2)$-gon is $\frac{1}{k+1}\binom{2k}{k}$, which is also called* **the k^{th} Catalan number**.

Proof. Let b_{n-1} be the number of triangulations of the n-gon, for $n \geq 3$. It is not hard to see that $b_2 = 1, b_3 = 2, b_4 = 5$. We want to find a general formula of b_n.

Consider the triangle T in a triangulation of n-gon which contains corners 1 and 2. The triangle T should contain a third corner, say i, where $3 \leq i \leq n$. We have the following two cases.

Case 1. If $i = 3$ or n, the triangle T divides the n-gon into the triangle T itself plus a $(n-1)$-gon, which results in b_{n-2} triangulations of n-gon.

Case 2. For $4 \leq i \leq n-1$, the triangle T divides the n-gon into three regions: a $(n-i+2)$-gon, triangle T and a $(i-1)$-gon, therefore it results in $b_{i-2} \times b_{n-i+1}$ many triangulations of n-gon.

Therefore, combining Cases 1 and 2, we get that

$$b_{n-1} = b_{n-2} + \sum_{i=4}^{n-1} b_{i-2}b_{n-i+1} + b_{n-2} = b_{n-2} + \sum_{j=2}^{n-3} b_j b_{n-j-1} + b_{n-2}.$$

By letting $b_0 = 0$ and $b_1 = 1$, we get

$$b_{n-1} = \sum_{j=0}^{n-1} b_j b_{n-1-j} \quad \text{for} \quad n \geq 3 \qquad \text{or} \qquad b_k = \sum_{j=0}^{k} b_j b_{k-j} \quad \text{for} \quad k \geq 2.$$

Let $f(x) = \sum_{k\geq 0} b_k x^k$. Note that $f^2(x) = \sum_{k\geq 0}\left(\sum_{j=0}^{k} b_j b_{k-j}\right)x^k$. Therefore

$$f(x) = x + \sum_{k\geq 2} b_k x^k = x + \sum_{k\geq 2}\left(\sum_{j=0}^{k} b_j b_{k-j}\right)x^k = x + \sum_{k\geq 0}\left(\sum_{j=0}^{k} b_j b_{k-j}\right)x^k = x + f^2(x).$$

Solving $f^2(x) - f(x) + x = 0$, we get that $f(x) = \frac{1+\sqrt{1-4x}}{2}$ or $\frac{1-\sqrt{1-4x}}{2}$. But notice that $f(0) = 0$, so it has to be the case that

$$f(x) = \frac{1-\sqrt{1-4x}}{2}.$$

Next, we apply the Newton's binomial theorem to get that

$$f(x) = \frac{1}{2} - \frac{1}{2}\sum_{k\geq 0}\binom{\frac{1}{2}}{k}(-4x)^k = \sum_{k\geq 1}\frac{(-1)^{k+1}4^k}{2}\binom{\frac{1}{2}}{k}x^k.$$

After plugging the obtained expression of $\binom{\frac{1}{2}}{k} = \frac{(-1)^{k-1}2}{4^k}\cdot\frac{(2k-2)!}{k!(k-1)!}$, we get that

$$f(x) = \sum_{k\geq 1}\frac{(2k-2)!}{k!(k-1)!}x^k = \sum_{k\geq 1}\frac{1}{k}\binom{2k-2}{k-1}x^k.$$

Note that $f(x)$ is the generating function of $\{b_k\}$, therefore

$$b_k = \frac{1}{k}\binom{2k-2}{k-1}.$$

This finishes the proof. ∎

3 Random walks

Consider a real axis with integer points $(0, \pm 1, \pm 2, \pm 3, \cdots)$ marked. A frog leaps among the integer points according to the following rules:

(1). At beginning, it sits at 1.

(2). In each coming step, the frog leaps either by distance 2 to the right (from i to $i+2$), or by distance 1 to the left (from i to $i-1$), each of which is randomly chosen with probability $\frac{1}{2}$ independently of each other.

Problem 3.1. *What is the probability that the frog can reach "0"?*

Solution. In each step, we use "+" or "−" to indicate the choice of the frog that is either to leap right or leap left. Then the probability space Ω can be viewed as the set of infinite vectors, where each entry is in $\{+, -\}$.

Let A be the event that the frog reaches 0. Let A_i be the event that the frog reaches 0 at the i^{th} step for the first time. So $A = \cup_{i=1}^{+\infty} A_i$ is a disjoint union. So $P(A) = \sum_{i=1}^{+\infty} P(A_i)$.

To compute $P(A_i)$, we can define a_i to be the number of trajectories (or vectors) of the first i steps such that the frog starts at 1 and reaches 0 at the i^{th} step for the first time. So

$$P(A_i) = \frac{a_i}{2^i}.$$

Then,

$$P(A) = \sum_{i=1}^{+\infty} \frac{a_i}{2^i}.$$

Let $f(x) = \sum_{i=0}^{+\infty} a_i x^i$ be the generating function of $\{a_i\}_{i \geq 0}$, where $a_0 := 0$. Thus,

$$P(A) = \sum_{i=1}^{+\infty} \frac{a_i}{2^i} = f\left(\frac{1}{2}\right).$$

We then turn to find the expression of $f(x)$.

Let b_i be the number of trajectories of the first i steps such that the frog starts at "2" and reaches "0" at the i^{th} step for the first time.

Let c_i be the number of trajectories of the first i steps such that the frog starts at "3" and reaches "0" at the i^{th} step for the first time.

First we express b_i in terms of $\{a_j\}_{j \geq 1}$. Since the frog only can leap to left by distance 1, if the frog can successfully jump from "i" to "0" in i steps, then this frog must reach "1" first. Let j be the number of steps by which the frog reaches "1" for the first time. So there are a_j trajectories from "2" to "1" at the j^{th} step for the first time.in the remaining $i-j$ steps the frog must jump from "1" to "0" and reach "0" at the coming $(i-j)^{th}$ step for the first time, so there are a_{i-j} trajectories that the frog can finish in exactly $i-j$ steps. In total,

$$b_i = \sum_{j=1}^{i-1} a_j a_{i-j}.$$

As $a_j = 0$,

$$b_i = \sum_{j=0}^{i} a_j a_{i-j}.$$

$$\Rightarrow \sum_{i\geq 0} b_i x^i = (\sum_{i\geq 0} a_i x^i)^2 = f^2(x).$$

Similarly, if we count the number c_i of trajectories from 3 to 0, we can obtain that

$$c_i = \sum_{j=0}^{i} a_j b_{i-j}.$$

$$\Rightarrow \sum_{i\geq 0} c_i x^i = \left(\sum_{i\geq 0} b_i x^i\right)\left(\sum_{i\geq 0} a_i x^i\right) = f^3(x).$$

Let us consider a_i from another point of view. After the first step, either the frog reaches "0" directly (if it leaps to left, so $a_1 = 1$), or it leaps to "3". In the latter case, the frog needs to jump from "3" to "0" using $i-1$ steps. Thus for $i \geq 2$, $a_i = c_{i-1}$.

Combining the above facts, we have

$$f(x) = \sum_{i=0}^{+\infty} a_i x^i = x + \sum_{i\geq 2} a_i x^i = x + \sum_{i\geq 2} c_{i-1} x^i = x + x\left(\sum_{j=0}^{+\infty} c_j x^j\right) = x + x\cdot f^3(x).$$

Let $a := P(A) = f(1/2)$. Then $a = \frac{1}{2} + \frac{a^3}{2}$, i.e., $(a-1)(a^2+a-1) = 0$, implying that

$$a = 1,\ \frac{\sqrt{5}-1}{2},\ \text{or}\ \frac{-\sqrt{5}-1}{1}.$$

Since $P(A) \in [0,1]$, we see $P(A) = 1$ or $\frac{\sqrt{5}-1}{2}$.

Note that $f(x) = x + xf^3(x)$. Consider the inverse function of $f(x)$, that is, $g(x) := \frac{x}{1+x^3}$. Consider the figure of $g(x)$. We find that $g(x)$ is increasing around $\frac{\sqrt{5}-1}{2}$ but decreasing around 1. Since $f(x) = \sum a_i x^i$ is increasing, $g(x)$ also increases. Thus it doesn't make sense for $g(x)$ being around $x = 1$. This explains that $P(A) = \frac{\sqrt{5}-1}{2}$. ∎

4 Exponential Generating Functions

Let $\mathbb{N}, \mathbb{N}_e$ and $\mathbb{N}_o$ be the sets of non-negative integers, non-negative even integers and non-negative odd integers, respectively.

Given n sets I_j of non-negative integers for $j \in [n]$, let $f_j(x) = \sum_{i\in I_j} x^i$. Let a_k be the number of integers solutions to $i_1 + i_2 + ... + i_n = k$, where $i_j \in I_j$. Then $\prod_{j=1}^{n} f_j(x)$ is the ordinary generating function of $\{a_k\}_{k\geq 0}$.

Problem 4.1. *Let S_n be the number of selections of n letters chosen from an unlimited supply of a's, b's and c's such that both of the numbers of a's and b's are even.*

Solution. We can write S_n as

$$S_n = \sum_{e_1+e_2+e_3=n,\ e_1,e_2\in\mathbb{N}_e,\ e_3\in\mathbb{N}} 1.$$

Using the previous fact, we see that $S_n = [x^n]f$, where

$$f(x) = \left(\sum_{i\in\mathbb{N}_e} x^i\right)^2 \left(\sum_{j\in\mathbb{N}} x^j\right) = \left(\frac{1}{1-x^2}\right)^2 \cdot \frac{1}{1-x}.$$

∎

Problem 4.2. *Let T_n be the number of arrangements (or words) of n letters chosen from an unlimited supply of a's, b's and c's such that both of the numbers of a's and b's are even. What is the value of T_n?*

Solution. To solve this, we define a new kind of generating functions.

Definition 4.3. *The exponential generating function for the sequence $\{a_n\}_{n\geq 0}$ is the power series*

$$f(x) = \sum_{n=0}^{\infty} a_n \cdot \frac{x^n}{n!}.$$

Then we have the following fact.

Fact 4.4. *If we have n letters including x a's, y b's and z c's (i.e. $x+y+z=n$), then we can form $\frac{n!}{x!y!z!}$ distinct words using them.*

Therefore, a selection (say x a's, y b's and z c's) can contribute $\frac{n!}{x!y!z!}$ arrangements to T_n. This implies that

$$T_n = \sum_{e_1+e_2+e_3=n,\ e_1,e_2\in\mathbb{N}_e,\ e_3\in\mathbb{N}} \frac{n!}{e_1!e_2!e_3!}.$$

Similar to defining the above $f(x)$ for S_n, we define the following for T_n. Let

$$g(x) := \left(\sum_{i\in\mathbb{N}_e} \frac{x^i}{i!}\right)^2 \left(\sum_{j\in\mathbb{N}} \frac{x^j}{j!}\right).$$

Claim. We have

$$[x^n]g = \frac{T_n}{n!}.$$

Proof. To see this, we expand $g(x)$. Then the term x^n in $g(x)$ becomes

$$\sum_{\substack{e_1+e_2+e_3=n,\\ e_1,e_2\in\mathbb{N}_e,\ e_3\in\mathbb{N}}} \frac{x^{e_1}}{e_1!}\cdot\frac{x^{e_2}}{e_2!}\cdot\frac{x^{e_3}}{e_3!} = \left(\sum_{\substack{e_1+e_2+e_3=n,\\ e_1,e_2\in\mathbb{N}_e,\ e_3\in\mathbb{N}}} \frac{n!}{e_1!e_2!e_3!}\right)\frac{x^n}{n!} = T_n\cdot\frac{x^n}{n!}.$$

So $[x^n]g = \frac{T_n}{n!}$, i.e., $g(x)$ is the exponential generating function of $\{T_n\}$. This finishes the proof of Claim. ∎

Using Taylor series: $e^x = \sum_{j\geq 0} \frac{x^j}{j!}$ and $e^{-x} = \sum_{j\geq 0}(-1)^j \frac{x^j}{j!}$, we have

$$\frac{e^x + e^{-x}}{2} = \sum_{j\in\mathbb{N}_e} \frac{x^j}{j!} \quad \text{and} \quad \frac{e^x - e^{-x}}{2} = \sum_{j\in\mathbb{N}_o} \frac{x^j}{j!}.$$

By the previous fact, we get

$$g(x) = \left(\frac{e^x + e^{-x}}{2}\right)^2 \cdot e^x = \frac{e^{3x} + 2e^x + e^{-x}}{4} = \sum_{n\geq 0} \left(\frac{3^n + 2 + (-1)^n}{4}\right) \cdot \frac{x^n}{n!}.$$

Therefore, we get that

$$T_n = \frac{3^n + 2 + (-1)^n}{4}.$$

∎