# Combinatorics

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## 1 Exponential Generating Function

Recall that the *exponential generating function* for the sequence $\{a_n\}_{n\geq 0}$ is the power series

$$f(x)=\sum_{n=0}^{+\infty} a_n \cdot \frac{x^n}{n!}.$$

As we shall see, ordinary generation functions can be used to find the number of selections; while exponential generation functions can be used to find the number of arrangements or some combinatorial objects **involving ordering**. We summarize this as the following facts.

**Fact 1.1.** *Given $I_j \subseteq \mathbb{N}^+$ for $j\in[n]$, let $f_j(x)=\sum_{i\in I_j} x^i$. And let $a_k=\sum_{\substack{i_1+\ldots+i_n=k,\\ i_j\in I_j}} 1$. Then*

$$\prod_{j=1}^{n} f_j(x)=\sum_{k=0}^{+\infty} a_k x^k.$$

**Fact 1.2.** *Given $I_j \subseteq \mathbb{N}^+$ for $j\in[n]$, let $g_j(x)=\sum_{i\in I_j} \frac{x^i}{i!}$. And let $b_k=\sum_{\substack{i_1+\ldots+i_n=k,\\ i_j\in I_j}} \frac{k!}{i_1!i_2!\ldots i_n!}$. Then*

$$\prod_{j=1}^{n} g_j(x)=\sum_{k=0}^{+\infty} \frac{b_k}{k!} x^k.$$

**Fact 1.3.** *Let $f(x)=\prod_{j=1}^{n} f_j(x)$. Then*

$$[x^k]f=\sum_{\substack{i_1+\ldots+i_n=k,\\ i_j\geq 0}} \prod_{j=1}^{n}[x^{i_j}]f_j.$$

**Fact 1.4.** *Let $f(x)=\prod_{j=1}^{n} f_j(x)$ and let $f_j(x)=\sum_{k=0}^{+\infty}\frac{a_k^{(j)}}{k!}x^k$. Then*

$$f(x)=\sum_{k=0}^{+\infty}\frac{A_k}{k!}x^k.$$

*if and only if*

$$A_k=\sum_{\substack{i_1+\ldots+i_n=k,\\ i_j\geq 0}} \frac{k!}{i_1!i_2!\ldots i_n!}\Big(\prod_{j=1}^{n} a_{i_j}^{(j)}\Big).$$

**Exercise 1.** *Find the number $a_n$ of ways to send $n$ students to 4 different classrooms (say $R_1$, $R_2$, $R_3$, $R_4$) such that each room has at least 1 students.*

*Solution.*

$$a_n = \sum_{\substack{i_1+i_2+i_3+i_4=n,\\ i_j\geq 1}} \frac{n!}{i_1!i_2!i_3!i_4!}.$$

Let $I_j = \{1, 2, ...\}$ for $j \in [4]$ and $g_j(x) = \sum_{i\geq 1} \frac{x^i}{i!} = e^x - 1$. By Fact 1.2, we have that

$$g_1g_2g_3g_4 = \sum_{n=0}^{+\infty} \frac{a_n}{n!}x^n = (e^x - 1)^4 = e^{4x} - 4e^{3x} + 6e^{2x} - 4e^x + 1.$$

Thus $a_n = 4^n - 4 \cdot 3^n + 6 \cdot 2^n - 4$ for $n \geq 4$. ∎

**Exercise 2.** *Let $a_n$ be the number of arrangements of type $A$ for a group of $n$ people, and let $b_n$ be the number of arrangements of type $B$ for a group of $n$ people.*

Define a new arrangement of $n$ people called type $C$ as follows:

- Divide the $n$ people into 2 groups (say $1^{st}$ and $2^{nd}$).
- Then arrange the $1^{st}$ group by an arrangement of type $A$, and arrange the $2^{nd}$ group by an arrangement of type $B$.

Let $c_n$ be the number of arrangements of type $C$ of $n$ people. Let $A(x), B(x), C(x)$ be the exponential generation function for $\{a_n\}, \{b_n\}, \{c_n\}$ respectively. Prove that $C(x) = A(x)B(x)$.

*Proof.* We can easily see that

$$c_n = \sum_{\substack{i+j=n,\\ i,j\geq 0}} \frac{n!}{i!j!} a_i b_j.$$

Then by Fact 1.4, $C(x) = A(x)B(x)$. ∎

## 2 Part II Basic of Graphs

In this second part of our course, we will discuss many interesting results in graph theory. We first introduce several basic definitions about graphs.

**Definition 2.1.** *A graph $G = (V, E)$ consists of a vertex set $V$ and an edge set $E$, where the elements of $V$ are called vertices and the elements of $E \subseteq \binom{V}{2} = \{(x, y) : x, y \in V\}$ are called edges.*

• If $E$ contains unordered pairs, then $G$ is an undirected graph, otherwise $G$ is a directed graph.

• In this couse, all graphs are undirected and simple, i.e., it has NO loops or muliple edges.

- We say vertices $x$ and $y$ are adjacent if $(x, y) \in E$, write $x \sim_G y$ or $x \sim y$ or $xy \in E$.

- We say the edge $xy$ is incident to the endpoints $x$ and $y$.

- Let $e(G)$ be the number of edges in G, i.e., $e(G) = |E(G)|$.

- The degree of a vertex $v$ in $G$, denoted by $d_G(v)$, is the number of edges in $G$ incident to $v$.

- The neighborhood of a vertex $v$ is the set of vertices $u$ that is adjacent to $v$, i.e., $N_G(v) = \{u \in V(G) : u \sim v\}$. Thus we have $d_G(v) = |N_G(v)|$.

- A graph $G' = (V', E')$ is a subgraph of $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E \cap \binom{V'}{2} \Leftrightarrow G' \subseteq G$.

- A subgraph $G' = (V', E')$ of $G = (V, E)$ is induced, if $E' = E \cap \binom{V'}{2}$.

**Definition 2.2.** *Two graphs $G = (V, E)$ and $G' = (V', E')$ are isomorphic if there exists a bijection $f : V \to V'$ such that $i \sim_G j$ if and only if $f(i) \sim_{G'} f(j)$.*

- A graph on $n$ vertices is a complete graph (or a clique), denoted by $K_n$, if all pairs of vertices are adjacent. So we have $e(K_n) = \binom{n}{2}$.

- A graph on $n$ vertices is called an independent set, denoted by $I_n$, if it contains no edges at all.

- Given a graph $G = (V, E)$, its complement is a graph $\overline{G} = (V, E^c)$ with $E^c = \binom{V}{2} \backslash E$.

- The degree sequence of a graph $G = (V, E)$ is a sequence of degrees of all vertices listed in a non-decreasing order.

- The path $P_k$ of length $k - 1$ is a graph $v_1 v_2 ... v_k$ where $v_i \sim v_{i+1}$ for $i \in [k-1]$. Note that the length of a path $P$ (denoted by $|P|$) is the number of *edges* in $P$.

- A cycle $C_k$ of length $k$ is a graph $v_1 v_2 ... v_k v_1$ where $v_i \sim v_{i+1}$ for $i \in [k]$, where $v_{k+1} = v_1$.

- A graph $G$ is planar, if we can draw $G$ on the plane such that its intersects only at their endpoints.

**Exercise 3.** *Show that $K_4$ is planar but $K_5$ is not.*

The following Handshaking Lemma is the most basic lemma in graph theory.

**Lemma 1** (Handshaking Lemma)**.** *In any graph $G = (V, E)$,*

$$\sum_{v \in V} d_G(v) = 2e(G).$$

*Proof.* Let $F = \{(e, v) : e \in E(G), v \in V(G)$ such that $v$ is adjacent to $e\}$. Then

$$\sum_{e \in E(G)} 2 = |F| = \sum_{v \in V} d_G(v).$$

∎

**Corollary 1.** *In any graph $G$, the number of vertices with odd degree is even.*

*Proof.* Let $O = \{v \in V(G) : d(v) \text{ is odd}\}$ and $\mathcal{E} = \{v \in V(G) : d(v) \text{ is even}\}$. Then by Lemma 1,

$$2e(G) = \sum_{v \in O} d_G(v) + \sum_{v \in \mathcal{E}} d_G(v).$$

Thus we have $\sum_{v \in O} d_G(v)$ is even, moreover we have $|O|$ is even. ∎

**Corollary 2.** *In any graph $G$, if there exists a vertex with odd degree, then there are at least two vertices with odd degree.*

## 3 Sperner's Lemma

Let us consider the following application of Corollary 2. First we draw a triangle in the plane, with 3 vertices $A_1A_2A_3$. Then we divide this triangle $\triangle = A_1A_2A_3$ into small triangles such that no triangle can have a vertex inside an edge of any other triangle. Then we assign 3 colors (say 1,2,3) to all vertices of these triangles, under the following rules.

(1) The vertex $A_i$ is assigned by color i for $i \in [3]$.

(2) All vertices lying on the edge $A_iA_j$ of the large triangle are assigned by the color $i$ or $j$.

(3) All interior vertices are assigned by any color 1,2,3.

**Lemma 2** (Sperner's Lemma (a planar version))**.** *For any assignment of colors described as above, there always exists a small triangle whose three vertices are assigned by three colors* $1, 2, 3$.

*Proof.* Define an auxiliary graph $G$ as following.

- Its vertices are the faces of small triangles and the outer face. Let $z$ be the vertex representing the outer face.
- Two vertices of $G$ are adjacent, if the two corresponding faces are neighboring faces and the two endpoints of their common edge are colored by 1 and 2.

We consider the degree of any vertex $v \in V(G)\backslash\{z\}$.

(1) If the face of $v$ has NO two endpoints with color 1 and 2, then $d_G(v) = 0$.

(2) If the face of $v$ has 2 endpoints with color 1 and 2. Let $k$ be the color of the third endpoint of this face. If $k \in \{1, 2\}$, then $d_G(v) = 2$. Otherwise $k = 3$, then $d_G(v) = 1$ and this triangle has 3 colors 1,2,3.

Thus we have that $d_G(v)$ is odd if and only if $d_G(v) = 1$, and then the face of $v$ has colors 1,2,3. Now we consider $d_G(z)$ and claim that it must be odd. Indeed, the edge of $G$ incident to $z$ obviously have to go across $A_1A_2$. Consider the sequence of the colors of the endpoints on $A_1A_2$, from $A_1$ to $A_2$. Then $d_G(z) = \#$ of alternations between 1 and 2 in this sequence, which must be odd. By Corollary 2, since the graph $G$ has a vertex $z$ with odd degree, there must be another vertex $v \in V(G)\backslash\{z\}$ with odd degree. Then $d(v) = 1$ and the face of $v$ has colors 1,2,3. ∎

**Theorem 3.1** (Brouver's Fixed Point Theory in 2-dimension)**.** *Every continuous function $f$ : $\triangle \to \triangle$ has a fixed point $x$, that is, $f(x) = x$.*

*Proof.* Consider a sequence of refinements of $\triangle$. Define three auxiliary functions $\beta_i : \triangle \to R$ for $i \in \{1, 2, 3\}$ as following:

For any $a = (x, y) \in \triangle$,

$$\begin{cases} \beta_1(a) & = x, \\ \beta_2(a) & = y, \\ \beta_3(a) & = 1 - x - y. \end{cases}$$

For any continuous $f : \triangle \to \triangle$, define $M_i = \{a \in \triangle : \beta_1(a) \geqslant \beta_1(f(a))\}$ for $i \in \{1, 2, 3\}$. Then we have the following facts.

(1) Any point $a \in \triangle$ belongs to at least one $M_i$.

(2) If $a \in M_1 \cap M_2 \cap M_3$, then $a$ is a fixed point.

We want to define a coloring $\phi : \triangle \to \{1, 2, 3\}$ such that

(a) Any $a \in \triangle$ with $\phi(a) = i$ belongs to $M_i$.

(b) The coloring $\phi$ satisfies the conditions of Sperner's Lemma for any subdivision of $\triangle$.

Next we show such $\phi$ exists. This is because

- For the point $A_i$ (say $i = 1$), we have that $A_1 = (1, 0) \in M_1$, so we can let $\phi(A_i) = i$.
- Consider a vertex $a = (x, y) \in A_1A_2$, i.e., $x + y = 1$. Then $a \in M_1 \cup M_2$, otherwise $x + y < 1$ which is a contradiction. So we can color $a$ by 1 or 2.

Now we define a sequence $\{\triangle_1, \triangle_2, ...\}$ of subdivisions of $\triangle$ such that the maximum diameter of small triangles in $\triangle_n$ is going to 0 as $n \to +\infty$. Applying Sperner's Lemma to each $\triangle_n$ and the coloring $\phi$, we get that there exists a small triangle $A_1^{(n)}A_2^{(n)}A_3^{(n)}$ in $\triangle_n$ which has 3 colors 1,2,3.

Consider the sequence $\{A_1^{(n)}\}_{n\geq 1}$. Since everything is bound, there is a subsequence $\{A_1^{(n_k)}\}_{k\geq 1}$ such that $\lim\limits_{k\to+\infty} A_1^{(n_k)} = p \in \triangle$ exists. Since the diameter of $A_1^{(n)}A_2^{(n)}A_3^{(n)}$ is going to be 0 as $n \to +\infty$, we see that $\lim\limits_{k\to+\infty} A_2^{(n_k)} = \lim\limits_{k\to+\infty} A_3^{(n_k)} = p$. Since $\beta_i(A_i^{(n_k)}) \geqslant \beta_i(f(A_i^{(n_k)}))$ for $i \in [3]$ and f is continuous. We get $\beta_i(p) = \lim\limits_{k\to+\infty} \beta_i(A_i^{(n_k)}) \geq \lim\limits_{k\to+\infty} \beta_i(f(A_i^{(n_k)})) = \beta_i(f(p))$ for $i \in [3]$. This implies that $p \in M_1 \cap M_2 \cap M_3$, so $p$ is a fixed point of $f$, i.e., $f(p) = p$. ∎