Combinatorics

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1 Lecture 6. Trees

Definition 1.1. *A graph G is connected, if for any vertices u and v, G contains a path from u to v. Otherwise, we say G is disconnected.*

Definition 1.2. *A component of a graph G is a maximal connected subgraph of G.*

Definition 1.3. *A graph T is called a tree if it is connected but contains no cycles. A vertex in a tree T with degree one is called a leaf.*

Fact 1.4 (Euler's Formula on trees). *For any tree $T = (V, E)$, we have $|V| = |E| + 1$.*

Proof. First, any tree has at least one leaf. As otherwise, all vertices have degree at least 2, then this gives a cycle, a contradiction.

Next we apply induction on n. Consider the base case that $n = 2$, the tree is an edge, then we are done. Now we assume the statement holds for any tree on $n-1$ vertices. Consider a tree T on n vertices ($n \geq 2$). We know that T contains a leaf, call v. It is easy to see that $T - \{v\}$ is still a tree as it is connected and has no cycles which has $n-1$ vertices. By induction, $T - \{v\}$ has $n-2$ edges. So T has $n-1$ edges. ∎

Fact 1.5. *Any tree T with at least 2 vertices has at least 2 leaves.*

Proof. Assume for a contradiction that an n-vertex tree T has exactly one leaf v, then $d(u) \geq 2$ for any $u \in V(T)\backslash\{v\}$. Thus

$$2(n-1) = 2e(T) = \sum_{x \in V(T)} d(x) \geq 2(n-1) + 1 = 2n - 1,$$

a contradiction. ∎

Theorem 1.6 (Tree characterization). *Let $T = (V, E)$ be a graph. Then the following are equivalent:*

(i). T is a tree (i.e. connected and no cycle).

(ii). T is a "minimal" connected graph. (i.e. deleting any edge will result in a disconnected graph.)

(iii). T is a "maximal" graph without a cycle. (i.e. adding any new edge will result in a cycle.)

Proof. (i)⇒(ii): Suppose (ii) fails, then there exists $e = xy \in E(T)$ such that $T - \{e\}$ is still connected. Then $T - \{e\}$ has a path P from x to y. So $P \cup \{e\}$ is a cycle in T, a contradiction.

(ii)⇒(i): Suppose (i) fails, then T contains a cycle C. If we delete any edge e from C, $T - \{e\}$ remains connected, a contradiction.

(i)⇒(iii): For any new edge $e = xy$, as T is connected, T has a path P from x to y. Thus, $P \cup \{e\}$ gives a cycle.

(iii)⇒(i): Suppose (i) fails, so T is disconnected. Then T has two components, say D_1 and D_2. Pick $x \in D_1$ and $y \in D_2$. If we add the new edge $e = xy$, then it is easy to see that $T + \{e\}$ still has no cycle, a contradiction. ∎

Definition 1.7. *Given a graph G, a subgraph H of G is a spanning subgraph if $V(H) = V(G)$.*

Fact 1.8. *Any graph G is connected if and only if it contains a spanning tree.*

Proof. If G has a spanning tree then it is connected.

Suppose G is connected. Deleting edges of G until it satisfies the property (ii) in the theorem-1.6, then we get a spanning tree. ∎

Definition 1.9. *Given a connected graph G with n vertices, say $v_1, ..., v_n$. Let $ST(G)$ be the number of labelled spanning trees in G.*

Theorem 1.10 (Cayley's Formula)**.** *For an integer $n \geq 2$,*

$$ST(K_n) = n^{n-2}.$$

We will give 3 proofs for this formula.

1.1 The first proof of Cayley's formula

Let $V(K_n) = \{v_1, v_2, \cdots, v_n\}$ and given a spanning tree T. Then

$$\sum_{i=1}^{n} d(v_i) = 2e(T) = 2n - 2.$$

Now we introduce a lemma.

Lemma 1.11. *Let $d_1, d_2, ..., d_n$ be positive integers with $\sum_{i=1}^{n} d_i = 2n - 2$. Then the number of spanning trees in K_n on vertex set $\{v_1, ..., v_n\}$ satisfying $d(v_i) = d_i$ is equal to*

$$\frac{(n-2)!}{(d_1 - 1)!(d_2 - 1)! \cdots (d_n - 1)!}.$$

Proof. We prove by induction on n. Base case is trivial. When $n = 2$, $d_1 = d_2 = 1$. There is only one spanning tree.

Now we assume that this statement holds for any sequence of $n - 1$ positive integers. Then consider $d_1, ..., d_n$ with $\sum_{i \in [n]} d_i = 2n - 2$. By average, $(\sum d_i)/n < 2$, so there exists some $d_i = 1$, say $d_n = 1$. Let $\mathcal{F}$ be the family of all spanning trees with $d(v_i) = d_i$ for $i \in [n]$. And let $\mathcal{F}_i = \{T - \{v_n\} : T \in \mathcal{F}$, the unique neighbor of v_n in T is $v_i\}$. So $|\mathcal{F}| = \sum_{i=1}^{n-1} |\mathcal{F}_i|$. All trees in $\mathcal{F}_i$ have $n - 1$ vertices $\{v_1, v_2, \cdots, v_{n-1}\}$ such that

$$\begin{cases} d(v_j) = d_j & j \neq i \\ d(v_i) = d_i - 1 & \text{otherwise.} \end{cases}$$

By induction, we have

$$|\mathcal{F}_i| = \frac{(n-3)!}{(d_1-1)!\cdots(d_i-2)!\cdots(d_{n-1}-1)!} = \frac{(n-3)!(d_i-1)}{\prod_{j=1}^{n-1}(d_j-1)!}.$$

So

$$|\mathcal{F}| = \sum_{i=1}^{n-1}|\mathcal{F}_i| = \frac{(n-3)!}{\prod_{j=1}^{n-1}(d_j-1)!}\left(\sum_{i=1}^{n-1}(d_i-1)\right) = \frac{(n-2)!}{\prod_{j=1}^{n}(d_j-1)!}.$$

∎

Recall the multinomial Theorem:

$$(x_1+x_2+\cdots x_k)^n = \sum_{i_1+\cdots i_k=n}\frac{n!}{i_1!\cdots i_k!}x_1^{i_1}\cdots x_k^{i_k},$$

which implies

$$k^n = \sum_{i_1+\cdots i_k=n}\frac{n!}{i_1!\cdots i_k!}.$$

Thus we have

$$ST(K_n) = \sum_{\substack{\sum_{i=1}^n d_i=2n-2\\ d_i\geq 1}}\frac{(n-2)!}{\prod_{j=1}^{n}(d_j-1)!} = n^{n-2}.$$

1.2 The second proof of Cayley's formula

Definition 1.12. *A digraph $D=(V,A)$ consists of a vertex set V and an arc set $A\subseteq\{(i,j): i,j\in V\}$*

Let $\mathscr{D}$ be the family of digraphs $D=([n],A)$ such that each vertex in D has exactly one arc going out (i.e. each vertex has out degree one).

Fact 1.13.

$$|\mathscr{D}| = n^n.$$

Proof. Consider the set $\mathscr{F}=\{\text{all mapping } f:[n]\to[n]\}$. It is easy to see there exists a bijection between $\mathscr{D}$ and $\mathscr{F}$. So $|\mathscr{D}|=|\mathscr{F}|=n^n$. ∎

Definition 1.14. *Given a spanning tree of K_n, we choose 2 special vertices (one marked by a circle and the other marked by a square; these two vertices can be the same vertex). We call such a subject (the spanning tree with 2 special vertices) as a vertebrate.*

Let $\mathscr{V}$ be a family of all vertebrates on $[n]$. Clearly, $|\mathscr{V}|=ST(K_n)n^2$. So to get the Cayley's formula, it suffices to show $|\mathscr{V}|=n^n$.

Lemma 1.15. *There exists a bijection between $\mathscr{V}$ and $\mathscr{D}$.*

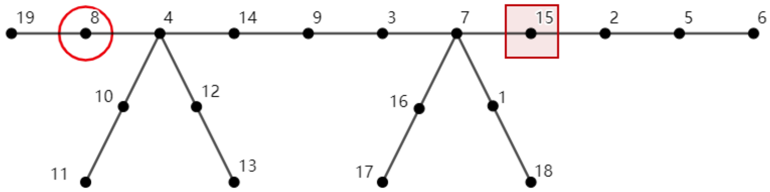

Figure 1: A vertebrate

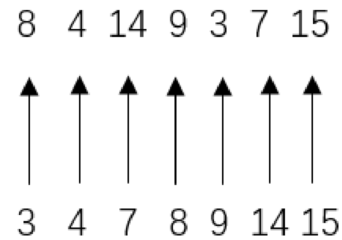

Figure 2: D_1

Proof. Consider a $W \in \mathscr{V}$ (see figure 1). Let P be the unique path in W between the two special vertices (marked by a circle and a square); and view P as a directed path from the circle to the square.

We then define a digraph D_1 on $V(P)$ by assign the following arcs (figure 2): that is, we place two rows, where the 1^{st} row is from P and the 2^{nd} row is the increasing sequence of $V(P)$, then we orient the arcs of D_1 from the vertices of the 2^{nd} row to the one above it. Thus each vertex in D_1 has exactly one arc out and one arc going in.

Exercise 1.1. *D_1 consists of vertex-disjoint directed cycle. (possibly loops and 2-cycles)*

Next, we extend D_1 to a digraph D on $[n]$, by the following:

(1) We remove all edges of P from W.

(2) Then $W - E(P)$ consists of subtrees, each having one vertex from $V(P)$. We direct the edges of these subtrees such that they point to the unique vertex of the component contained in $V(P)$.

(3) There arcs product in (2) together with the arcs of D_1, define a new graph D_W on $[n]$. This should be easy to see that $D_W \in \mathscr{D}$.

So we just define a mapping $\varphi : \mathscr{V} \to \mathscr{D}$, by assigning $\varphi(W) = D_W,\ W \in \mathscr{D}$. Next, We show φ is a bijection.

Step 1. We can define $\varphi^{-1} : \mathscr{D} \to \mathscr{V}$ such that $\varphi^{-1} \cdot \varphi = Id$.

Remark: In any D_W, $V(D_1)$ consists of all vertices in D_W contained in a directed cycle.

Take any $D \in \mathscr{D}$, there exists some vertex of D contained in a directed cycle. Let X be the set of all such vertices of D. Since $D[X]$ consists of vertex-disjoint directed cycles, there is a nature way to define a path as following (see figure 3):

First, list the vertices of X in the increasing order. Second, list the out-neighbor vertices of X in another row, respectively. Then the second row defines a path P be the special path in the vertebrate. Then it is easy to define the rest part of the vertebrate say W. So we have $D \in \mathscr{D} \xrightarrow{\varphi^{-1}} W \in \mathscr{V}$. We can check that $\varphi^{-1} \cdot \varphi = Id$.

Step 2. φ is a surjective.

We have proved in Step 1 that for any $D \in \mathscr{D}$, there exists $W \in \mathscr{V}$ satisfying $\varphi(W) = D$.

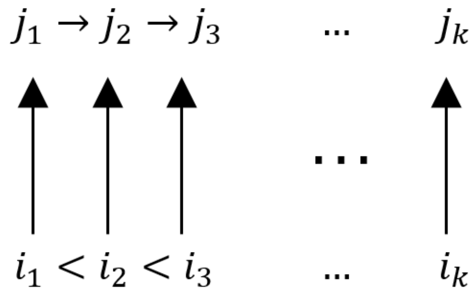

Figure 3: Define a path

Therefore indeed φ is a bijection.

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Combining Fact 1.13 with Lemma 1.15, we get $ST(K_n) = n^{n-2}$.