Combinatorics

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1 The third proof of Cayley's formula (using Linear Algebra)

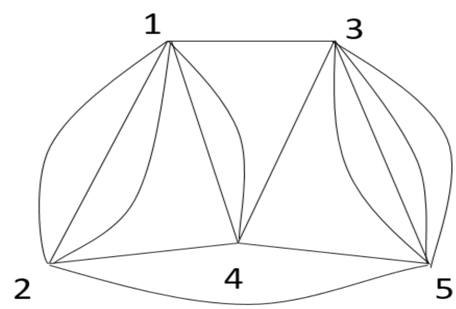
Definition 1.1. *A multigraph is a graph, where we allow multiple edges between vertices but do not allow loops.*

For a multigraph G in $[n]$, we define the Laplace matrix $Q = (q_{ij})_{n\times n}$ of G as follows:

$$q_{ij} = \begin{cases} d_G(i), & \text{if } i = j. \\ -m, & \text{if } i \neq j \text{ and there are } m \text{ edges between } i \text{ and } j. \end{cases}$$

Note that Q is symmetric, and the sum of each row/column is 0.

For example



$$Q = \begin{pmatrix} 6 & -3 & -1 & -2 & 0 \\ -3 & 5 & 0 & -1 & -1 \\ -1 & 0 & 6 & -1 & -4 \\ -2 & -1 & -1 & 5 & -1 \\ 0 & -1 & -4 & -1 & 6 \end{pmatrix}.$$

For an $n \times n$ matrix Q, let Q_{ij} be the $(n-1)\times(n-1)$ matrix obtained from Q by deleting the i^{th} row and j^{th} column.

Theorem 1.2. *For any multigraph G, $ST(G) = det(Q_{11})$, where Q_{ij} is the $(n-1)\times(n-1)$ matrix obtained from the Laplace matrix Q of G by deleting the i^{th} row and j^{th} column.*

Proof. We prove this by using induction on the number of edges in G. Base case, suppose that $e(G) = 1$. Then it holds trivially.

Now we consider a multigraph G and assume this holds for any multigraph with less than $e(G)$ edges. Take any edge e in G. Define two multigraph as following.

1. $G - e =$ the multigraph obtained from G be deleting the edge e.

2. G/e = the multigraph obtained from G by contracting the two endpoints x, y of e into a new vertex z and adding new edges in $\{zu : xu \in E(G)\} \cup \{zu : yu \in E(G)\}$.

Let Q' and Q'' be the Laplace matrices of $G - e$ and G/e respectively. If in a multigraph G the vertex number 1 is not incident to any edge, then we have $T(G) = 0$. The first row of the Laplace matrix consists only of zeros, the sum of the rows of Q_{11} is also zero. Thus, $det(Q_{11}) = 0$. If the vertex number 1 is incident to at least one edge. More precisely, assume that the edge e has endpoints 1 and 2. So

$$Q' = \begin{pmatrix} 5 & -2 & -1 & -2 & 0 \\ -2 & 4 & 0 & -1 & -1 \\ -1 & 0 & 6 & -1 & -4 \\ -2 & -1 & -1 & 5 & -1 \\ 0 & -1 & -4 & -1 & 6 \end{pmatrix}, Q'' = \begin{pmatrix} 5 & -1 & -3 & -1 \\ -1 & 6 & -1 & -4 \\ -3 & -1 & 5 & -1 \\ -1 & -4 & -1 & 6 \end{pmatrix}.$$

Let $Q_{11,22}$ be the matrix obtained from Q by deleting the first two rows and the first two columns. Then we have

$$det(Q_{11}) = det((Q')_{11}) + det(Q_{11,22}). \tag{1.1}$$

We also see that

$$Q_{11,22} = (Q'')_{11}. \tag{1.2}$$

By (1.1) and (1.2) we have

$$det(Q_{11}) = det((Q')_{11}) + det((Q'')_{11}). \tag{1.3}$$

Claim. For any edge e in G, we have

$$ST(G) = ST(G - e) + ST(G/e). \tag{1.4}$$

Proof. We divide the spanning trees of G into two classes:

-the 1^{st} class contains those spanning trees of G NOT containing e, which are exactly $ST(G - e)$.

-the 2^{nd} class contains those spanning trees of G containing e. We can easily see that the trees in the 2^{nd} class are one-to-one corresponding to the spanning trees of G/e.

This proves (1.4)

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By induction, we have $ST(G - e) = det(Q'_{11})$, $ST(G/e) = det((Q'')_{11})$. By (1.3), we have $ST(G) = det(Q_{11})$.

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Proof of Cayley's Formula. For K_n, we have

$$Q = \begin{pmatrix} n-1 & -1 & \cdots & -1 \\ -1 & n-1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & n-1 \end{pmatrix}_{n \times n},$$

which implies that $ST(G) = det(Q_{11}) = n^{n-2}$. ∎

2 Intersecting Family

Definition 2.1. *A family $\mathcal{F} \subset 2^{[n]}$ is intersecting if for any $A, B \in \mathcal{F}$, we have $A \bigcap B \neq \emptyset$.*

Fact 2.2. *For any intersecting family $\mathcal{F} \subset 2^{[n]}$, we have $|\mathcal{F}| \leq 2^{n-1}$.*

Proof. Consider all pairs $\{A, A^c\}$ for all $A \subset [n]$. Note that there are exactly 2^{n-1} such pairs, and $\mathcal{F}$ can have at most one subset from every pairs. This proves $|\mathcal{F}| \leq 2^{n-1}$. ∎

Note that this is tight:

- $\mathcal{F} = \{A \subset [n] : 1 \in A\}$.
- For n is odd, $\mathcal{F} = \{A \in [n] : |A| > \frac{n}{2}\}$.

A harder problem: What is the largest intersecting family $\mathcal{F} \subset \binom{[n]}{k}$, for fixed k?

Theorem 2.3 (Erdős-Ko-Rado Theorem)**.** *For $n \geq 2k$, the largest intersecting family $\mathcal{F} \subset \binom{[n]}{k}$ has size $\binom{n-1}{k-1}$.*

Moreover, if $n > 2k$, then the largest intersecting family $\mathcal{F} \subset \binom{[n]}{k}$ must be $\mathcal{F} = \{A \in \binom{[n]}{k} : t \in A\}$ for some fixed $t \in [n]$.

Proof. Take a cyclic permutation $\pi = (a_1, a_2, ..., a_n)$ of $[n]$. Note that there are $(n-1)!$ cyclic permutations of $[n]$ in total.

Let $\mathcal{F}_\pi = \{A \in \mathcal{F}$: A appears as k consecutive numbers in the circuit of π.$\}$

Claim 1. For all cyclic permutation π, assume $n \geq 2k$, then $|\mathcal{F}_\pi| \leq k$.

Proof. Pick $A \in \mathcal{F}_\pi$, say $A = \{a_1, a_2, ..., a_k\}$. We call the edges $a_n a_1, a_k a_{k+1}$ as the boundary edges of A, and the edges $a_1a_2, a_2a_3, ..., a_{k-1}a_k$ as the inner-edges of A. We observe that for any distinct $A, B \in \mathcal{F}_\pi$, the boundary-edges of A and B are distinct. For any $B \in \mathcal{F}_\pi \setminus \{A\}$, as $A \bigcap B \neq \phi$. we see that one of the boundary-edges of B must be an inner-edge of A. But A has $k-1$ inner-edges, so we see that there are at most $k-1$ many subsets in $\mathcal{F}_\pi \setminus \{A\}$. So $|\mathcal{F}_\pi| \leq k$. ∎

Next we do a double-counting. Let N be the number of pairs (π, A), where π is a cyclic permutation of [n], and $A \in \mathcal{F}_\pi$. By Claim 1, $N = \sum_\pi |\mathcal{F}_\pi| \leq k(n-1)!$. Fix A, how many cyclic π such that $A \in \mathcal{F}_\pi$? The answer is $k!(n-k)!$. So the number of cyclic permutations π such that π contains the elements of A as k consecutive numbers is $k!(n-k)!$. So we have

$$k(n-1)! \geq N = \sum_{A \in \mathcal{F}} k!(n-k)! = |\mathcal{F}|k!(n-k)!,$$

which implies that

$$|\mathcal{F}| \leq \frac{k \cdot (n-1)!}{k!(n-k)!} = \binom{n-1}{k-1}.$$

If $n > 2k$, for the extremal case $\mathcal{F} = \binom{n-1}{k-1}$, we want to show $\mathcal{F}$ must be a star. From the preview proof, we see that for any cycle permutation π, $|\mathcal{F}_\pi| = k$. And we have following claim.

Claim 2. Fix any $\pi = (a_0, a_1, ..., a_{n-1})$. If $\mathcal{F}_\pi = \{A_1, A_2, ..., A_k\}$, then $A_1 \cap A_2 \cap ... \cap A_k = \{t\}$ for some $0 \leq t \leq n-1$, where $A_j = \{a_{j+r}, a_{j+r+1}, ..., a_{j+r+k-1}\}$ for $1 \leqslant j \leqslant k$ and for some $0 \leq r \leq n-1$ (where the indices are taken under the additive group $\mathbb{Z}_n$.) .

Proof. With loss of generality, suppose that $A = \{a_1, ...a_k\} \in \mathcal{F}_\pi$. From the preview proof, we know $a_i a_{i+1}$ is boundary-edge of some $B_i \in \mathcal{F}_\pi$ where $i \in [k-1]$, and for any distinct $A, B \in \mathcal{F}_\pi$, the boundary-edges of A and B are distinct. For any $C \in \mathcal{F}$, we color the two boundary-edges by 1 and 0, respectively, according to the clockwise direction. Since $a_0 a_1$ has color 1 and $a_k a_{k+1}$ has color 0. There must exist $\ell \in [k]$ such that $a_{\ell-1} a_\ell$ has color 1 and $a_\ell a_{\ell+1}$ has color 0. Let $A_1 = \{a_{\ell-k+1}, a_{\ell-k+2}, ..., a_{\ell-1}, a_\ell\}$ and $A_k = \{a_\ell, a_{\ell+1}, ..., a_{\ell+k-1}\}$. Since $\mathcal{F}$ is intersecting and $n > 2k$, there dose not exist j such that $a_{j-1} a_j$ has color 0 and $a_j a_{j+1}$ has color 1. Then $a_{\ell-1+i}, a_{\ell+i}$ has color 0 for every $i \in [k]$. This finishes the proof. ∎

Fix π, let $\mathcal{F}_\pi = \{A_1, A_2, ..., A_k\}$ and let $A_1 \cap A_2 \cap ... \cap A_k = \{t\}$. If any element of $\mathcal{F}$ contains t, then $\mathcal{F}$ is a star, we are done. So we may assume that there exists $A_0 \in \mathcal{F}$ such that $t \notin A_0$.

Claim 3. For any $B \in \binom{A_1 \cup A_k \setminus \{t\}}{k-1}$, we have $B \cup \{t\} \in \mathcal{F}$.

Proof of Claim 3. Consider another cycle permutation $\pi^{'}$ with A_1, A_k unchanged, but the order of the integers inside $A_1 \setminus \{t\}$ and $A_k \setminus \{t\}$ are changed.

Since $A_1, A_k \in \mathcal{F}_{\pi'}$, by Claim 2 all other k-sets in $A_1 \cup A_k$ formed by k consecutive integers on $\pi^{'}$ are also in $\mathcal{F}_{\pi'} \subseteq \mathcal{F}$. Repeating using the argument, we prove Claim 3. ∎

Claim 4. The subset $A_0 \in \mathcal{F}$ (with $t \notin A_0$) satisfies $A_0 \subseteq A_1 \cup A_k \setminus \{t\}$.

Proof of Claim 4. Otherwise, A_0 has at most $k-1$ elements in $A_1 \cup A_k$. Then we have $|A_1 \cup A_k - A_0| \geqslant k$ (as $|A_1 \cup A_k| = 2k-1$). So, we can pick a k-subset $B \subseteq A_1 \cup A_k - A_0$ such that $t \in B$. By Claim 3, we have $B \in \mathcal{F}$. But $A_0 \cap B = \emptyset$, contradicting that $\mathcal{F}$ is intersecting. This proves Claim 4. ∎

Claim 5. We have $\binom{A_1 \cup A_k}{k} \subseteq \mathcal{F}$.

Proof of Claim 5. Consider any $i \in A_0$, let $B_i = (A_1 \cup A_k \setminus A_0) \cup \{i\}$. Since $t \in B_i$, by Claim 3, we have $B_i \in \mathcal{F}$. Repeating the proof of Claim 3, we can obtain that any k-subset of $A_1 \cup A_k$ containing i belongs to $\mathcal{F}$. In other words, any k-subset B of $A_1 \cup A_k$ must intersect A_0, and thus belongs to $\mathcal{F}$. Then we have $\binom{A_1 \cup A_k}{k} \subseteq \mathcal{F}$. ∎

If there exists a k-subset $C \in \mathcal{F}$ such that $B \nsubseteq A_1 \cup A_k$, then $|A_1 \cup A_k - B| \geqslant k$. So there exists $D \subseteq A_1 \cup A_k - C$ with $|D| = k$. By Claim 5, we have $D \in \mathcal{F}$, but $C \cap D = \emptyset$, a contradiction. This proves $\binom{A_1 \cup A_k}{k} = \mathcal{F}$.

Since $n > 2k$, we see $|\mathcal{F}| = \binom{2k-1}{k} = \binom{2k-1}{k-1} < \binom{n-1}{k-1} = |\mathcal{F}|$, a contradiction. This completes the proof.

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