Combinatorics

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1 The Second Proof of Erdős-Ko-Rado Theorem

Definition 1.1. *A Kneser graph $K(n,k)$ with $n \geqslant 2k$ is a graph with vertex set $\binom{[n]}{k}$ such that for any two sets $A, B \in \binom{[n]}{k}$, A is adjacent to B in $K(n,k)$ if and only if $A \cap B = \varnothing$.*

One can easily check that $K(5,2)$ is the Petersen graph.

Definition 1.2. *Given a graph G, we let $\alpha(G)$ be the number of vertices in a largest independent set in G.*

We note that any independent set in $K(n,k)$ is an intersecting family in $\binom{[n]}{k}$. Therefore, we have the following.

Theorem 1.3 (Erdős-Ko-Rado (Restatement))**.** *For $n \geq 2k$, $\alpha(K(n,k)) \leqslant \binom{n-1}{k-1}$.*

Definition 1.4. *The adjacency matrix $A_G = (a_{ij})_{n\times n}$ of an n-vertex graph G is defined by*

$$a_{ij} = \begin{cases} 1, & \text{if } ij \in E(G), \\ 0, & \text{otherwise.} \end{cases}$$

Definition 1.5. *The eigenvalues $\lambda_1 \geqslant \lambda_2 \geqslant ... \geqslant \lambda_n$ of A_G is called the eigenvalues of G. The eigenvectors $\vec{v}_1, \vec{v}_2, ..., \vec{v}_n$ of A_G satisfying*

$$\begin{cases} A_G\vec{v}_i = \lambda_i\vec{v}_i, \\ ||\vec{v}_i|| = 1, \\ \vec{v}_i \perp \vec{v}_j \quad \text{for any } i \neq j, \end{cases}$$

are called the orthonormal eigenvectors of G.

Note that A_G is an $n \times n$ 0/1 symmetric matrix. Thus all the eigenvalues of G are real numbers.

Definition 1.6. *A graph G is d-regular if all vertices have the same degree d.*

Exercise 1.7. *If G is d-regular, then the largest eigenvalue of G is d.*

Theorem 1.8 (Hoffman's Theorem)**.** *If an n-vertex graph G is d-regular with eigenvalues $\lambda_1 \geqslant \lambda_2 \geqslant ... \geqslant \lambda_n$, then $\alpha(G) \leqslant n \cdot \dfrac{-\lambda_n}{\lambda_1 - \lambda_n}$*

Proof. Let $V(G)=[n]$. Let $\vec{v}_1,...,\vec{v}_n$ be the corresponding orthonormal eigenvectors of eigenvalues $\lambda_1 \geqslant \lambda_2 \geqslant ... \geqslant \lambda_n$ of G. Thus we have

$$\begin{cases} A_G\vec{v}_i = \lambda_i \vec{v}_i, \\ ||\vec{v}_i|| = 1, \\ \vec{v}_i \perp \vec{v}_j \Leftrightarrow < \vec{v}_i, \vec{v}_j >= 0, \ \forall i \neq j. \end{cases}$$

Let I be an independent set of G with $|I| = \alpha(G)$. Let $\vec{1}_I \in \{0,1\}^n$ be the vector such that its j^{th} coordinate is 1 if $j \in I$, and 0 otherwise. Then we can write

$$\vec{1}_I = \sum_{i=1}^{n} \alpha_i \vec{v}_i \ \text{ for some } \alpha_i \in \mathbb{R}.$$

Then we have

$$|I| =< \vec{1}_I, \vec{1}_I >=< \sum_i \alpha_i \vec{v}_i, \sum_j \alpha_j \vec{v}_j >= \sum_{i=1}^{n} \alpha_i^2, \tag{1.1}$$

where $\alpha_i =< \vec{1}_I, \vec{v}_i >$.

Since G is d-regular, we have that $\lambda_1 = d$ and $\vec{v}_1 = (1/\sqrt{n}, ..., 1/\sqrt{n})^T$. So we get

$$\alpha_1 =< \vec{1}_I, \vec{v}_1 >= \frac{|I|}{\sqrt{n}}. \tag{1.2}$$

Since I is an independent set in G,

$$\vec{1}_I^T A_G \vec{1}_I = \sum_{i,j} (\vec{1}_I)_i a_{ij} (\vec{1}_I)_j = 0,$$

where $A(G) = (a_{ij})$. On the other hand, we also have

$$\begin{aligned} 0 = \vec{1}_I^T A_G \vec{1}_I &= \big(\sum_i \alpha_i \vec{v}_i\big)^T A_G \big(\sum_j \alpha_j \vec{v}_j\big) = \big(\sum_i \alpha_i \vec{v}_i\big)^T \big(\sum_j \alpha_j \lambda_j \vec{v}_j\big) \\ &= \sum_{i=1}^{n} \alpha_i^2 \lambda_i \geq \alpha_1^2 \lambda_1 + (\alpha_2^2 + \cdots + \alpha_n^2)\lambda_n \stackrel{\text{by (1.1) (1.2)}}{=} \frac{|I|^2}{n}\lambda_1 + \left(|I| - \frac{|I|^2}{n}\right)\lambda_n. \end{aligned}$$

Thus we have

$$\frac{|I|^2}{n}\lambda_1 + \left(|I| - \frac{|I|^2}{n}\right)\lambda_n \leq 0, \quad \text{and} \quad |I|\left(\frac{|I|}{n}\lambda_1 + \lambda_n - \frac{|I|}{n}\lambda_n\right) \leq 0,$$

which implies that

$$\alpha(G) = |I| \leq n \cdot \frac{-\lambda_n}{\lambda_1 - \lambda_n}.$$

∎

Theorem 1.9 (see GTM 207, Theorem 9.4.3)**.** *The eigenvalues of Kneser graph $K(n,k)$ are:*

$$u_j = (-1)^j \binom{n-k-j}{k-j} \text{ of multiplicity } \binom{n}{j} - \binom{n}{j-1}$$

for every $0 \leq j \leq k$.

Proof of Theorem 1.3. Consider the eigenvalues of $K(n,k)$, say $\lambda_1 \geq \lambda_2 \cdots \geq \lambda_{\binom{n}{k}}$, where $\lambda_1 = \binom{n-k}{k}$, $\lambda_{\binom{n}{k}} = -\binom{n-k-1}{k-1}$. By Hoffman's bound,

$$\alpha(K(n,k)) \leq \binom{n}{k} \frac{-\lambda_{\binom{n}{k}}}{\lambda_1 - \lambda_{\binom{n}{k}}} = \binom{n}{k} \frac{\binom{n-k-1}{k-1}}{\binom{n-k}{k} + \binom{n-k-1}{k-1}} = \binom{n-1}{k-1},$$

as desired. ∎

2 Partially Ordered Sets (Poset)

Let X be a finite set.

Definition 2.1. *R is a relation on X, if $R \subseteq X \times X$ where $X \times X$ denote the Cartesion product of X, i.e., $X \times X = \{(x_1, x_2) : \forall x_1, x_2 \in X\}$. If $(x,y) \in R$, then we often write xRy.*

Definition 2.2. *A partially ordered set (poset for short) is an ordered pair (X, R), where X is a finite set and R is a relation on X such that the following hold:*

(1) R is reflective: xRx for any $x \in X$,

(2) R is antisymmetric: if xRy and yRx, then $x = y$,

(3) R is transitive: if xRy and yRz, then xRz.

Example 2.3. *Consider the poset $(2^{[n]}, \subseteq)$, where "$\subseteq$" denotes the inclusion relationship.*

We often use "$\preccurlyeq$" to replace the use of "R". So poset $(X,R) = (X, \preccurlyeq)$ and $xRy = x \preccurlyeq y$. If $x \preccurlyeq y$ but $x \neq y$, then $x \prec y$, and we say x is a predecessor/child of y.

Definition 2.4. *Let $(X, \preccurlyeq)$ be a poset. We say an element x is an immediate predecessor of y, if*

(1) $x \prec y$,

(2) there is no element $t \in X$ such that $x \prec t \prec y$.

In this case, we write $x \lhd y$.

Fact 2.5. *For $x, y \in (X, \preccurlyeq)$, $x \prec y$ if and only if there exist $z_1, z_2, ..., z_k \in X$ such that $x \lhd z_1 \lhd z_2 \lhd ... \lhd z_k \lhd y$. (Note that here k can be 0, i.e., $x \lhd y$.)*

Proof. ($\Leftarrow$) This direction is trivial, by transitive property.

($\Rightarrow$) Let $x \prec y$. Let $M_{xy} = \{t \in X : x \prec t \prec y\}$. We prove by induction on $|M_{xy}|$.

Base case is clear, if $|M_{xy}| = 0$, then $x \lhd y$. Now we may assume $M_{xy} \neq \emptyset$ and the statement holds for any $u \prec v$ with $|M_{uv}| < n$. Suppose $x \prec y$ with $|M_{xy}| = n \geqslant 1$. Pick any $t \in M_{xy}$ and consider M_{xt} and M_{ty}. Clearly $M_{xt} \subsetneq M_{xy}$ and $M_{ty} \subsetneq M_{xy}$ (because of transitive property). By induction on M_{xt} and M_{ty}, there exist $x_1, x_2, ..., x_m \in X$ and $y_1, y_2, ..., y_l \in X$ such that $x \lhd x_1 \lhd x_2 \lhd ... \lhd x_m \lhd t$ and $t \lhd y_1 \lhd y_2 \lhd ... \lhd y_l \lhd y$. Thus, $x \lhd x_1 \lhd x_2 \lhd x_m \lhd t \lhd y_1 \lhd ... \lhd y_l \lhd y$ and we are done. ∎

Now we can express a poset in a diagram.

Definition 2.6. *The Hassa diagram of a poset $(X, \preccurlyeq)$ is a drawing in the plane such that*

(1) each element of X is drawn as a nod in the plane,

(2) each pair $x \lhd y$ is connected by a line segment,

(3) if $x \lhd y$, then the nod x must appear lower in the plane then the nod y.

The fact that $x \prec y$ if and only if $x \lhd x_1 \lhd x_2 \lhd ... \lhd x_k \lhd y$ now can be restated as follows: $x \prec y$ if and only if we can find a path in the Hassa diagram from nod x to nod y, strictly from bottom to top.

Definition 2.7. *Let $(X_1, \preccurlyeq_1)$ and $(X_2, \preccurlyeq_2)$ be two posets. A mapping $f : X_1 \to X_2$ is called an embedding of $(X_1, \preccurlyeq_1)$ in $(X_2, \preccurlyeq_2)$ if*

(1) f is injective,

(2) $f(x) \preccurlyeq_2 f(y)$ if and only if $x \preccurlyeq_1 y$.

Theorem 2.8. *For every poset $(X, \preccurlyeq)$ there exists an embedding of $(X, \preccurlyeq)$ in $\mathscr{B}_X = (2^X, \subseteq)$.*

Proof. Consider the mapping $f : X \to 2^X$ by letting $f(x) = \{y \in X : \; y \preccurlyeq x\}$ for any $x \in X$. It suffices to verify that f is an embedding of $(X, \preccurlyeq)$ in $(2^X, \subseteq)$.

Firstly, f is injective. If $f(x) = f(y)$ for $x, y \in X$, then $x \in f(x) = f(y)$ and $x \preccurlyeq y$. Similarly we have $y \preccurlyeq x$. So $x = y$.

Secondly, $f(x) \subseteq f(y)$ if and only if $x \preccurlyeq y$. To see this, if $x \preccurlyeq y$, then clearly $f(x) \subseteq f(y)$. Now suppose $f(x) \subseteq f(y)$. Since $x \in f(x) \subseteq f(y)$, we have $x \preccurlyeq y$. This shows that f indeed is an embedding. ∎

Definition 2.9. *Let $P = (X, \preccurlyeq)$ be a poset.*

(1) For distinct $x, y \in X$, if $x \prec y$ or $y \prec x$, then we say that x, y are comparable; otherwise, x, y are incomparable.

(2) The set $A \subseteq X$ is an antichain of P, if any two elements in A are incomparable. Let $\alpha(P)$ be the maximum size of an antichain of P

(3) The set $B \subseteq X$ is a chain of P, if any two elements of B are comparable. Let $\omega(P)$ be the maximum size of a chain of P

Consider the Hassa diagram, $\omega(P)$ means the maximum number of vertices in a path (from bottom to top) in this diagram. So $\omega(P)$ is also called the height of P and $\alpha(P)$ is called the width of P.

Definition 2.10. *An element $x \in X$ is minimal in $P = (X, \preccurlyeq)$, if x has no predecessor in P.*

Fact 2.11. *The set of all minimal elements of $P = (X, \preccurlyeq)$ forms an antichain of P.*

Theorem 2.12. *For any poset $P = (X, \preccurlyeq)$, $\alpha(P) \cdot \omega(P) \geq |X|$.*

Proof. We inductively define a sequence of posets $P_i = (X_i, \preceq)$ and a sequence of sets $M_i \subset P_i$, such that each M_i is the set of minimal elements of P_i, and $X_i = X - \sum_{j=0}^{i-1} M_j$, where $M_0 = \emptyset$.

First, set $P_1 = P = (X, \preccurlyeq), X_1 = X$ and $M_1 = \emptyset$. Assume posets $P_i = (X_i, \preccurlyeq)$ and M_{i-1} are defined for all $1 \leqslant i \leqslant k$. Let $M_i = \{$ all minimal elements of $P_i\}$ and let $X_{i+1} = X - M_1 \bigcup ... \bigcup M_i$. Then let P_{i+1} be the subposet of P restricted on X_{i+1}. We keep doing this until $X_{\ell+1} = \emptyset$. By Fact 2.11, each M_i is an antichain of P_i. Since P_i is the restricted subposet of P on X_i, M_i is also an antichain of P. So

$$|M_i| \leq \alpha(P).$$

It suffices to find a chain $x_1 \prec x_2 \prec ... \prec x_\ell$ in P, such that $x_i \in P_i = (X_i, \preccurlyeq)$ for $i \in [\ell]$. Indeed, if this holds, then

$$X = M_1 \bigcup M_2 \bigcup ... \bigcup M_\ell \text{ and } |X| = \sum_{i=1}^{\ell} |M_i| \leq \alpha(P) \cdot \ell \leq \alpha(P) \cdot \omega(P).$$

In fact, by the definition of M_i, we can claim something stronger holds: For any $x \in M_i$ $(2 \leq i < \ell)$, there exists $y \in M_i$, such that $y \prec x$. This completes the proof. ∎