Combinatorics

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1 Poset

1.1 The order from disorder

Definition 1.1. *Consider a sequence $X = (x_1, x_2, ..., x_n)$ of n real numbers. A subsequence $(x_{i_1}, x_{i_2}, ..., x_{i_m})$ of X, where $i_1 < i_2 < ... < i_m$, is* monotone*, if either $x_{i_1} \leq x_{i_2} \leq ... \leq x_{i_m}$ or $x_{i_1} \geq x_{i_2} \geq ... \geq x_{i_m}$.*

For example, $(10, 9, 7, 4, 5, 1, 2, 3) \longrightarrow (10, 9, 7, 5, 1)$.

Theorem 1.2 (Erdős-Szekeres Theorem)**.** *For any sequence $(x_1, x_2, ..., x_{n^2+1})$ of length $n^2 + 1$, there exists a monotone subsequence of length $n + 1$.*

Proof. Let $X = [n^2 + 1]$. We define a poset $P = (X, \preceq)$ as following: $i \preceq j$ if and only if $i \leq j$ and $x_i \leq x_j$.

It is easy to check that $P = (X, \preceq)$ indeed defines a poset (reflective antisymmetric and transitive). By the previous result that $\alpha(P) \cdot w(P) \geq |X| = n^2 + 1$, we have either $w(P) \geq n + 1$ or $\alpha(P) \geq n + 1$.

Case 1. $w(P) \geq n + 1$.

There exists a chain of size $n + 1$, say $\{i_1, i_2, .., i_{n+1}\}$. By definition, $x_{i_1} \leq x_{i_2} \leq ... \leq x_{i_{n+1}}$ is an increasing subsequence of length $n + 1$.

Case 2. $\alpha(P) \geq n + 1$.

There exists an antichain of size n+1, say $\{i_1, i_2, .., i_{n+1}\}$. We may assume that $i_1 < i_2 < ... < i_{n+1}$ being antichain, it implies that $x_{i_1} > x_{i_2} > ... > x_{i_{n+1}}$ is a decreasing subsequence of $(x_1, x_2, ..., x_{n^2+1})$. ∎

Remark 1.3. *What we proved is a bit stronger: there is either an increasing subsequence of length $n + 1$ or a strictly decreasing subsequence of length $n + 1$.*

Exercise 1.4. *Find examples to show that Erdős-Szekeres Theorem is optimal: there exists a sequence of n^2 reals such that NO monotone subsequence of length $n + 1$.*

1.2 The Pigeonhole Principle

Theorem 1.5 (The Pigeonhole Principle)**.** *Let X be a set with at least $1 + \sum_{i=1}^{k}(n_i - 1)$ elements and let $X_1, X_2, ..., X_k$ be disjoint sets forming a partition of X. Then, there exists some i, such that $|X_i| \geq n_i$.*

(1) Two equal degrees.

Theorem 1.6. *Any graph has two vertices of the same degree.*

Proof. Let G be a graph with n vertices. Suppose that G does not have two vertices of same degree. So the only exceptional case will be that there is exactly one vertex of degree i for all $i \in \{0, 1, ..., n-1\}$. But this is impossible to have a vertex with degree 0 and a vertex with degree $n-1$ at the same time. ∎

Exercise 1.7. *For any n, find an n-vertex graph G, which has exactly two vertices with the same degree.*

(2) Subsets without divisors.

Question 1.8. *How large a subset $S \subset [2n]$ can be such that for any $i, j \in S$, we have $i \nmid j$ and $j \nmid i$?*

Obviously, we can take $S = \{n+1, n+2, ..., 2n\}$ with $|S| = n$.

Theorem 1.9. *For any $S \subset [2n]$ with $|S| \geq n+1$, there exist $i, j \in S$ such that $i|j$.*

Proof. For any odd integer $2k-1 \in [2n]$, define $S_{2k-1} = \{2^i \cdot (2k-1) \in S : i \geq 0\}$. Clearly, $S = \bigcup_{k=1}^{n} S_{2k-1}$. Since $|S| \geq n+1$, there exists some $|S_{2k-1}| \geq 2$ say $x, y \in S_{2k-1}$. It is easy to see that we have $x|y$ or $y|x$. ∎

(3) Rational approximation.

Theorem 1.10. *Given $n \in \mathbb{Z}^+$, for any $x \in \mathbb{R}^+$, there is a rational number $\frac{p}{q}$ such that $1 \leq q \leq n$ and $|x - \frac{p}{q}| < \frac{1}{nq}$.*

Proof. For any $x \in \mathbb{R}^+$, define $\{x\} = x - \lfloor x \rfloor$ be the fractional part of x. Consider $\{ix\} \in [0, 1)$, for any $i = 1, 2, ..., n+1$. Partition $[0, 1)$ into n subintervals $[0, \frac{1}{n}), [\frac{1}{n}, \frac{2}{n}), ..., [\frac{n-1}{n}, 1)$. By Pigeonhole Principle, there exists a subinterval $[\frac{k}{n}, \frac{k+1}{n})$ contains two reals say $\{ix\}$ and $\{jx\}$ for $1 \leq i < j \leq n+1$. Then we have $\{(j-i)x\} \in [0, \frac{1}{n}) \cup [1 - \frac{1}{n}, 1)$. Let $q = j - i \leq n$. So $\{qx\} \in (0, \frac{1}{n}) \cup [1 - \frac{1}{n}, 1)$, i.e. $qx = p + \epsilon$ for some $p \in \mathbb{Z}^+$ and $|\epsilon| < \frac{1}{n}$. Then we have $x = \frac{p}{q} + \frac{\epsilon}{q}$, which implies that $|x - \frac{p}{q}| = |\frac{\epsilon}{q}| < \frac{1}{nq}$. ∎

1.3 Second proof of Erdős-Szekeres Theorem

Theorem 1.11 (Erdős-Szekeres Theorem)**.** *For any sequence of $mn+1$ real numbers $\{a_0, a_1, ..., a_{mn}\}$, there is an increasing subsequence of length $m+1$ or a decreasing subsequence of length $n+1$.*

The second proof. Consider any sequence $\{a_0, a_1, ..., a_{mn}\}$. For any $i \in \{0, 1, ..., mn\}$, let f_i be the maximum size of an increasing subsequence starting at a_i. We may assume $f_i \in \{1, 2, ..., m\}$ for any $i \in \{0, 1, ..., mn\}$. By Pigeonhole Principle, there exists a $s \in \{1, 2, ..., m\}$ such that there are at least $n+1$ elements $i \in \{0, 1, ..., m\}$ satisfying $f_i = s$. Let these elements be $i_1 < i_2 < ... < i_{n+1}$.

We claim that $a_{i_1} \geq a_{i_2} \geq ... \geq a_{i_{n+1}}$. Why? If $a_{i_j} < a_{i_{j+1}}$ for some $j \in [n]$, then we would extend the max increasing subsequence of length s starting at $a_{i_{j+1}}$ by adding a_{i_j} to obtain an increasing subsequence starting at a_{i_j} of length s+1, a contradiction to $f_{i_j} = s$.

∎

2 Ramsey's Theorem

Fact 2.1 (A party of six). *Suppose a party has six participants. Participants may know each other or not. Then there must be three participants who know each other or do not know each other, i.e. any* 6*-vertex graph* G *has a* K_3 *or* I_3.

Proof. We consider a graph G on six vertices say $[6]$. Each vertex i represents one participant: i and j are adjacent if and only if they know each other. Then we need to show that there are three vertices in G which form a triangle K_3 or an independent set I_3.

Consider vertex 1. There are five other persons. So 1 is adjacent to three vertices or not adjacent to three vertices. By symmetry, we may assume that 1 is adjacent to three vertices, say $2, 3, 4$. If one of pairs $\{2, 3\}, \{2, 4\}, \{3, 4\}$ is adjacent, then we have a K_3. Otherwise, $\{2, 3, 4\}$ forms an independent set of size three. This finishes the proof. ∎

Definition 2.2. *An* r*-edge-coloring of* K_n *is a mapping* $f : E(K_n) \longrightarrow \{1, 2, ..., r\}$ *which assigns one of the colors* $1, 2, ..., r$ *to each edge of* K_n.

Definition 2.3. *Given an* r*-edge-coloring of* K_n*. A clique in* K_n *is called* monochromatic*, if all its edges are colored by the same color.*

Then the example of a party of six says that any 2-edge-coloring of K_6 has a monochromatic K_3.

Theorem 2.4 (Ramsey's Theorem (2-colors-version)). *Let* $k, \ell \geq 2$ *be any two integers. Then there exists an integer* $N = N(k, \ell)$*, such that any* 2*-edge-coloring of* K_N *(with colors red and blue) has a blue* K_k *or a red* K_ℓ.

Proof. We will prove by induction on $k + \ell$ that any blue/red-edge-coloring of a clique on $N = \binom{k+\ell-2}{k-1}$ vertices has a blue K_k or a red K_ℓ.

Base case is trivial (as we have $N = \binom{k+\ell-2}{k-1} = \ell$ where $k = 2$ and $N = \binom{k+\ell-2}{k-1} = k$ where $\ell = 2$).

We may assume that the statement holds for $k'+\ell' \leq k+\ell-1$. Let $N_1 = \binom{k+\ell-3}{k-2}$, $N_2 = \binom{k+\ell-3}{k-1}$, and $N = \binom{k+\ell-2}{k-1}$. So $N_1 + N_2 = N$.

Consider any red/blue-edge-coloring of K_N. Consider any vertex x. Let $A = \{y \in V(K_n) - \{x\} : \text{edge } xy \text{ is blue}\}$ and $B = \{y \in V(K_n) - \{x\} : \text{edge } xy \text{ is red}\}$. So $|A| + |B| = N - 1 = N_1 + N_2 - 1$. By Pigeonhole Principle we have either $|A| \geq N_1$ or $|B| \geq N_2$.

Case 1. $|A| \geq N_1 = \binom{(k-1)+\ell-2}{(k-1)-1}$.

The vertices of A contains a $K_{\binom{(k-1)+\ell-2}{(k-1)-1}}$ where edges are blue or red. By induction on this $K_{\binom{(k-1)+\ell-2}{(k-1)-1}}$ for the pair $\{k-1, \ell\}$, so A has a blue K_{k-1} or a red K_ℓ. In the former, by adding the vertex x to that blue K_{k-1}, we can obtain a blue K_k in the K_N.

Case 2.$|B| \geq N_2 = \binom{k+\ell-3}{k-1}$.

This case is similar (by induction on $\{k, \ell - 1\}$). ∎

Definition 2.5. *For* $k, \ell \geq 2$*, the* Ramsey Number $R(k, \ell)$ *denotes the smallest integer* N *such that any* 2*-edge-coloring of* K_N *has a blue* K_k *or a red* K_ℓ.

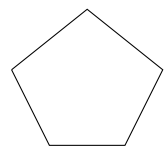
Remark 2.6. *Ramsay Theorem says that* $R(k, \ell) \leq \binom{k+\ell-2}{k-1}$.

Let us try to understand this definition a bit more:

- $R(k, \ell) \leq L$ if and only if any 2-edge-coloring of K_L has a blue K_k or a red K_ℓ.
- $R(k, \ell) > M$ if and only if there exists a 2-edge-coloring of K_M which has no blue K_k nor red K_ℓ.

Fact 2.7. *(1) $R(k, \ell) = R(\ell, k)$.*
(2) $R(2, \ell) = \ell$ and $R(k, 2) = k$.
(3) $R(3, 3) = 6$.

Proof. It is easy to know that (1) and (2) is right. We have $R(3, 3) \leq 6$ from the fact on a party of six. On the other hand, we have $R(3, 3) > 5$ form the following graph (if u, v are adjacent, we color edge uv blue, otherwise we color edge uv red).



∎

Exercise 2.8. $R(k, \ell) \leq R(k - 1, \ell) + R(k, \ell - 1)$.

Theorem 2.9. *If for some (k, ℓ), the numbers $R(k - 1, \ell)$ and $R(k, \ell - 1)$ are even, then*

$$R(k, \ell) \leq R(k - 1, \ell) + R(k, \ell - 1) - 1.$$

Proof. Let $n = R(k - 1, \ell) + R(k, \ell - 1) - 1$. So n is odd. Consider any 2-edge-coloring of K_n. For any vertex x, define the following as before $A_x = \{y : xy \text{ is blue}\}$ and $B_x = \{y : xy \text{ is red}\}$.

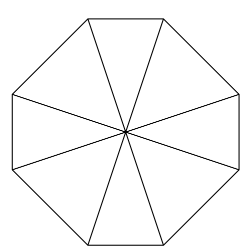
The previous proof tells us that if $|A_x| \geq R(k - 1, \ell)$ or $|B_x| \geq R(k, \ell - 1)$, then we can find a blue K_k or a red K_ℓ. Thus, we may assume that $|A_x| \leq R(k - 1, \ell) - 1$ and $|B_x| \leq R(k, \ell - 1) - 1$ for any vertex v, which implies that

$$n \leq A_x + B_x + 1 \leq R(k - 1, \ell) + R(k, \ell - 1) - 1$$

This shows that for each x, $|A_x| = R(k - 1, \ell) - 1$ and $|B_x| = R(k, \ell - 1) - 1$. Now we consider the graph G consisting of all blue edges. Note that G has an odd number of vertices and any vertex has odd degree. But this contradicts the Handshaking Lemma. ∎

Corollary 2.10. $R(3, 4) = 9$.

Proof. By the previous theorem, we have $R(3, 4) \leq R(2, 4) + R(3, 3) - 1 = 4 + 6 - 1 = 9$. On the other hand, we have $R(3, 4) > 8$ form the following graph (if u, v are adjacent, we color edge uv blue, otherwise we color edge uv red).



∎

Definition 2.11. *For any $k \geq 2$ and any integers $s_1, s_2, ..., s_k \geq 2$, the Ramsey number $R_k(s_1, s_2, ..., s_k)$ is the least integer N such that any k-edge-coloring of K_N has a clique K_{s_i} in color i, for some $i \in [k]$.*

Homework 2.12. $R_k(s_1, s_2, ..., s_k) < +\infty$.

Theorem 2.13 (Schur's Theorem)**.** *For $k \geq 2$, there exists some integer $N = N(k)$ such that for any coloring $\varphi : [N] \to [k]$, there exist three integers $x, y, z \in [N]$ satisfying that $\varphi(x) = \varphi(y) = \varphi(z)$ and $x + y = z$.*

Proof. Let $N = R_k(3, 3, ..., 3)$. Define a k-edge-coloring of K_N from the coloring φ as following: for any $i, j \in [N]$, define the color of ij to be $\varphi(|i - j|)$. By the definition of $R_k(3, 3, ..., 3)$, we can find a monochromatic triangle, say $ij\ell$. Suppose $i < j < \ell$, we have $\varphi(\ell - j) = \varphi(\ell - i) = \varphi(j - i)$. Let $x = \ell - j, y = \ell - i, z = j - i \in [N]$, we have $\varphi(x) = \varphi(y) = \varphi(z)$ and $x + y = z$. This finishes the proof. ∎

Remark 2.14. *It is also true to require x, y, z to be distinct, by considering $N = R_k(4, 4, ..., 4)$.*