Combinatorial Networks
Week 12, June 3-4

- **Theorem(Konig).** For bipartite G, $\chi'(G) = \Delta(G)$.
- **Theorem(Vizing).** For general G, $\Delta(G) \le \chi'(G) \le \Delta(G) + 1$.
- **Fact:** $\chi'_l(G) \ge \chi'(G) \ge \Delta(G)$.
- **Theorem(Kahn).** For general G, $\chi'_l(G) \le (1 + o(1))\chi'(G) \le \Delta(G) + o(\Delta(G))$.
- **Theorem(Dinitz \ Galvin).** For bipartite G, $\chi'_l(G) = \chi'(G) = \Delta(G)$. **Proof.** By Konig's Theorem, $\chi'(G) = \Delta(G)$.

  It suffices to show:

- **Theorem.** Given bipartite G of maximum degree Δ and list $\mathscr{L} = \{L_e\}_{e\in E(G)}$, where for each $|Le| = \Delta$, there is a legal coloring of $E(G)$ for the list.
- **Lemma.** Suppose H has an orientation D $s.t.$ for each $v \in V(H)$, $d^+_D(v) \le |L_v|$, andd every induced subgraph of D has a kernel. Then there is a legal coloring of $V(G)$ from the list $\{L_v\}_{v\in E(G)}$.

  Here, we work on $L(G)$, and want to achieve 2 goals,

  Goal 1:Find an orientation of $L(G)$ $s.t.$ $d^+_D(v) < \Delta$ for $\forall e \in V(L(G)) = E(G)$.

  Goal 2:Any induced subgraph of D has a kernel.

- **Theorem(Gale-Shopley).** For any bipartite G and set of preferences of $V(G)$, G has a stable matching.

  Define: For any $v \in V(H)$, its preference is a linear ordering on its neighbors.

  Define: For matching M, if a is matched then we use $M(a)$ to express the other end of the edge in M.

  Define: Given a matching M, a pair (a, b) is unstable, if

  · $(a, b) \in E(G)\backslash M$

  · a prefers b to it current partner $M(a)$

  · b prefers a to it current partner $M(b)$

  Define: A matching M its stable, if there is No unstable pairs.

  Proof: while $\exists a \in A s.t. L_a \ne \varnothing$. ($L_a$ is preference of a)

  − every $a \in A$ proposes to his top choice woman

  − each woman looks at her offers and tentatively takes the best offer (and reject the others)

  − each rejected man remove the rejecting woman from his preference.

  Once a man runs out of his preference, he leaves the game.

  By Konig's theorem: we can assume the edges of G have already been $\Delta-$colored, and we assume the coloring if $f : E(G) \to \{1, 2, \ldots, \Delta\}$.

Proof of Goal 1: Define orientation D of $L(G)$, let v_e, v'_e 2 adjacent vertices of $L(G)$, so $e, e' \in E(G)$ share a common vertex v.

if $v \in A$, we direct (v_e, v'_e), if $f(v_e) < f(v'_e)$

if $v \in B$, we direct (v_e, v'_e), if $f(v_e) > f(v'_e)$

One can verify that for $\forall v_e, d^+_D(v_e) \leq \Delta - 1 < \Delta$.

Fact: An independent set in $L(G)$ is a matching in G.

Proof of Goal 2: First define the preference for $L(G)$.

For any $a \in A$, $L_a = \{\cdots 3 > 2 > 1 \cdots\}$.

For any $b \in B$, $L_b = \{\cdots 3 < 2 < 1 \cdots\}$.

Let us check that for each $U \subseteq V(L(G))$, we have a kernel in U.

Let E_U be the set of corresponding edges in U from G and E_U induced a bipartite subgraph.

By G-S theorem, E_U has a stable matching M_u.

We show that M_U is a kernel of $D[U]$

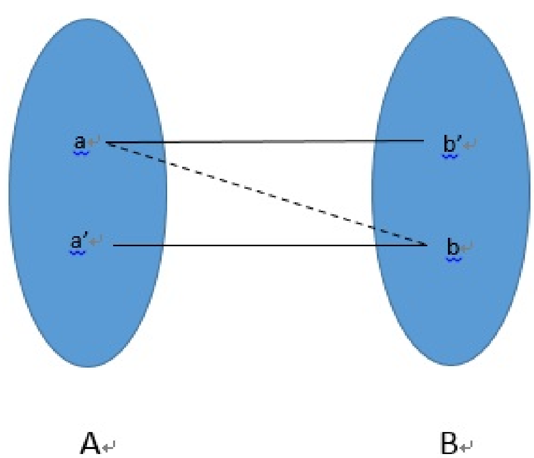
$\cdot$ M_U is independent (by Fact)

$\cdot$ Consider any edge $ab \in U \backslash M_U$

Since it can't be an unstable pair.

$\Rightarrow$ By definition of preference $f(ab') > f(ab)$ or $f(ba') < f(ba)$.

$\Rightarrow$ In D, $ab \to ab'$ or $ab \to a'b \Rightarrow$ kernel!



$L_a = \{\cdots b' > b \cdots\}, L_b = \{\cdots a' > a \cdots\}$

Exercise: if graph G is 2-connected and has a path of length $2S^2$, then G has a cycle of length$\geq 2s$.

- **Define:** G is k-critical if $\chi(G) = k$, but any proper subgraph $H \subsetneqq G$, $\chi C(H) < k$.

- **Fact:** Any k-critical subgraph is 2-connected.
- **Fact:** Any k-critical graph is (k-1)-edge-connected.
- **Theorem(Alon-Seymour).** if G is k-critical, then G has a path of length$\geq C \cdot \frac{\log n}{\log k}$ and a cycle of length$\geq C' \cdot \sqrt{\frac{\log n}{\log k}}$.

  $--$ Best Bound (Shapira-Thomas) such G has a cycle of length$\geq \frac{\log n}{100 \log k}$.

  $--$ (Gallai, 1963) $\exists$ a example k-critical $G s.t.$ the max cycle of G has length$\leq \frac{2(k-1)\log n}{\log(k-2)}$.

  Proof: Take a vertex v, and consider a DSF-tree T with root v. For any $u \in V(G)$, define $d(u)$ to be the depth of u, that is the number of edges in the path of T from u to the root v. For any e on T, define depth $d(e)$ to be j if e connect a vertex with depth j to a vertex with depth $j+1$.

  Claim: T has at most $k(k-1)^{j-1}$ edges with depth j.

  $''$claim$\Rightarrow T''$: let h be the height of DSF-tree T.

  $n-1 = \sum_{j=1}^{h} \#$ edges with depth$\leq \sum_{j=1}^{h} k(k-1)^{j-1} = k \cdot \frac{(k-1)^{h-1}-1}{k-1-1} \leq k^h$

  $\Rightarrow h \geq \frac{\log(n-1)}{\log k} \geq C \cdot \frac{\log n}{\log k}$

  so G has a path of length$\geq k \geq C \cdot \frac{\log n}{\log k}$

  Proof of claim: For $\forall e \in E(T)$, let f_e be the (k-1)-coloring on $G-e$, let $e = (v_d, v_{d-1})$ ($v_1 - v_2 - \cdots - v_d$ be the path of T from root to v_d) and let $F(e) \triangleq \big(f_e(v_1), f_e(v_2), \cdots, f_e(v_d)\big) \in [k-1]^d$

  We claim: if e and e' both have depth d, then $F(e) \neq F(e')$, suppose not, that $F(e) = F(e')$.

  $-$ Then, we can color $G - V(T_e)$ by using (k-1)-coloring f_e.

  $-$ Then, we can color $V(T_e)$ by using (k-1)-coloring $f_{e'}$.

  Now, (*)we check that the function combining by $f_{e'}|_{V(T_e)}$ and $f_e|_{G-V(T_e)}$ is a proper (k-1)-coloring of G.

  $$\begin{cases} f_e : (G - V(T_e)) \to [k-1] \\ f_{e'} : V(T_e) \to [k-1] \end{cases}$$

  The only edges from $V(T_e)$ to $V(G - T_e)$ are these from vertices in vTu, But $F(e) = F(e')$, so f_e and $f(e')$ coincide the colors on the vertices of vTu, Thus the combined function is a proper (k-1)-coloring, contradicting to $\chi(G) = k$. This proves claim(*) and thus claim.
- **Hard:** If G is 3-connected and has a path of length t, prove that G has a cycle of length ct.
- **Theorem(de-Bruiju-Erdos).** suppose G is an infinite graph, if any finite subgraph of G if k-colorable. Then G is also k-colorable.
- **Konig's Infinite lemma.** suppose $V_1, V_2, \ldots$ is an infinite sequence of finite sets; suppose $v \in V_{i+1}$ is connected to some vertex in V_i, Then there is an infinite path $v_1 \in V_1, v_2 \in V_2, \cdots, v_i \in V_i, \cdots$

Proof: There are infinite many paths ending at V_1, since V_1 is finite, there exists $a_1 \in V_1 s.t. \exists$ infinite paths ending at a_1, since V_2 is finite, $\exists a_2 \in V_2$ and infinite many paths passing through a_2 and ending at a_1

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Go on this argument, $\exists$ at least one infinite path passing through $v_1 \in V_1, v_2 \in V_2, \cdots, v_i \in V_i, \cdots$

Proof of Theorem: (Assume G is countable)

$V(G) = \mathbb{N}^+ = \{1, 2, \ldots\}$, let $G_i = G[\{1, 2, \ldots\}]$, let F_i be the set of all legal k-coloring of G_i, so F_i is nonempty and finite. We connect $f \in F_{i+1}$ and $g \in F_i$, if f agree with g on $\{1, 2, \ldots\}$.

$\Rightarrow$ Thus, each $f \in F_{i+1}$ connects to some vertices on F_i. By Konig's Infinite lemma, $\exists$ an infinite path passing through , $f_1 \in F_1, f_2 \in F_2, \cdots, f_i \in F_i, \cdots$

This gives a k-coloring of G!