Combinatorial Networks
Week 3, Wednesday

Ramsey's Theorem

- **Theorem.** If $R(s,t-1)$ and $R(s-1,t)$ are both even, then $R(s,t) \leq R(s,t-1) + R(s-1,t) - 1$.

- **Proof.** Let $n = R(s,t-1) + R(s-1,t) - 1$. Consider K_n with arbitrary 2-edge coloring. For any vertex v, define $B_v = \{y : vy \text{ is blue}\}$ and $R_v = \{y : vy \text{ is red}\}$. If there is some vertex v satisfying $|B_v| \geq R(s-1,t)$ or $|R_v| \geq R(s,t-1)$, then the conclusion follows immediately.
  Contrarily, if every vertex v satisfies $|B_v| < R(s-1,t)$ and $|R_v| < R(s,t-1)$, then
  $$n - 1 = |B_v| + |R_v| \leq R(s-1,t) + R(s,t-1) - 2.$$
  Since $n-1 = R(s,t-1)+R(s-1,t)-2$, it forces $|B_v| = R(s-1,t)-1$ and $|R_v| = R(s,t-1)-1$. Now note that n and $|B_v|$ are both odd and consider the subgraph induced by all the blue edges in K_n. It has odd vertices. Moreover, any vertex is of odd degree, but it's impossible! ∎

Schur's Theorem

- **Definition.** For integer $k \geq 2$ and integers $s_1, s_2, ..., s_k \geq 2$, the *Ramsey number* $R_k(s_1, ..., s_k)$ is the least integer N such that any k-edge coloring of K_N has a clique K_{s_i} in color i.

- **Exercise.** $R_k(s_1, ..., s_k) < \infty$.

- **Schur's Theorem.** For $k \geq 2$, there exists $N = N(k)$, such that for any k-edge coloring $c : [N] \to [k]$, we can find $x, y, z \in [N]$ satisfying $c(x) = c(y) = c(z)$ and $x + y = z$.

- **Proof.** Let $N = R_k(3, ..., 3)$. Given $c : [N] \to [k]$, define a k-edge coloring f on K_N by: $f(ij) = c(|i-j|)$, for all distinct $i, j \in [N]$.
  Now by the definition of Ramsey number $R_k(3, ..., 3)$, there is a monochromatic ijk. We may assume $i < j < k$, then $c(j-i) = c(k-i) = c(k-j)$. Write $x = j-i, z = k-i, y = k-j$, then $x + y = z$ and $c(x) = c(y) = c(z)$. ∎

- **Remark.** Schur used this to prove that the Fermat Last Theorem holds in $\mathbb{Z}_p$ for sufficiently large prime p. That is the following theorem.

- **Theorem.** For all integer $m \geq 1$, there is a prime number $p(m)$, such that for any prime $p \geq p(m)$, $x^p + y^p = z^p \ (mod\ p)$ has nontrivial solution.

- **Proof.** For prime p, consider the multiplicative group $\mathbb{Z}_p^*$.
  There is a $g \in \mathbb{Z}_p^*$ such that for any $x \in \mathbb{Z}_p^*$, one can write $x = g^{im+j}$ for some nonnegative integers i, j with $0 \leq j \leq m-1$.
  We define a m-coloring as follows:
  $$c : \mathbb{Z}_p^* \to [m] \ , \ x \mapsto j \in [m]$$

.

By Schur's Theorem, as long as p is large enough, there exist $x, y, z \in \mathbb{Z}_p^*$ such that $c(x) = c(y) = c(z)$ and $x + y = z$.
On the one hand, From $c(x) = c(y) = c(z)$ we can write $x = g^{i_1 m+j}, y = g^{i_2 m+j}, z = g^{i_3 m+j}$.
On the other hand, $x + y = z$ implies $g^{i_1 m} + g^{i_2 m} = g^{i_3 m}$. At last, denote $\tilde{x} = g^{i_1}, \tilde{y} = g^{i_2}, \tilde{z} = g^{i_3}$. Then $\tilde{x}, \tilde{y}, \tilde{z}$ is a nontrivial solution for the equation in $\mathbb{Z}_p$. ∎

Diagonal Ramsey number

- **Theorem.** Let integers n, s satisfying $\binom{n}{s}2^{1-\binom{s}{2}} < 1$, then $R(s, s) > n$.
- **Proof.** We look for a 2-edge coloring of K_n such that it has no monochromatic K_s. Consider a random 2-edge coloring of K_n, each edge is colored by blue or red both with probability $\frac{1}{2}$, independent of other edges. For any subset X of size s in [n], let A_X be the event that X is a monochromatic K_s, then $Pr(A_X) = 2(\frac{1}{2})^{\binom{s}{2}} = 2^{1-\binom{s}{2}}$, from which one can get
$$\sum_{|X|=s} Pr(A_X) = \binom{n}{s} 2^{1-\binom{s}{2}} > Pr(\bigcup_{|X|=s} A_X)$$
Thus $Pr(\text{there is no monochromatic } K_s) > 0$. That is, there is a 2-edge coloring of K_n with no monochromatic K_s. So $R(s, s) > n$. ∎
- **Corollary.**
$$R(s, s) \geq s2^{(s-1)/2}/e$$
.
- **Proof.** Let
$$n = (s/e)2^{(s-1)/2}(e/2)^{1/s} \geq s2^{(s-1)/2}/e.$$
Recall that
$$\binom{n}{s} < n^s/s!$$
and
$$e(s/e)^s \leq s!$$
Then
$$\binom{n}{s} < (ne/s)^s/e$$
Thus
$$\binom{n}{s} 2^{1-\binom{s}{2}} < (ne/s)^s 2^{1-\binom{s}{2}}/e = (e/s)^s (s/e)^s 2^{\binom{s}{2}} (e/2) 2^{1-\binom{s}{2}}/e = 1$$
By the theorem proved just before, we get
$$R(s, s) > n \geq s2^{(s-1)/2}/e.$$
∎