Combinatorial Networks
Week 6, April 22-23

Even-cycles-free

- For bipartite H (so $\chi(H) = 2$), it is unknown the order of $ex(n, H)$.
- For any H with $\chi(H) = k + 1 \geqslant 3$, $ex(n, H) = (1 - \frac{1}{k} + o_H(1))\frac{n^2}{2}$.
- For special bipartite graphs, the even cycles C_{2t}, it is known that $ex(n, C_{2t}) \leqslant O(n^{1+\frac{1}{t}})$
- **Conjecture.** $\forall t \geqslant 2$, $ex(n, C_{2t}) = \Theta(n^{1+\frac{1}{t}})$
- **Definition.** Let $\mathcal{F} = \{\text{some graphs } H\}$, $ex(n, \mathcal{F}) =$ the maximum of $e(G)$, G has n vertices, $\forall H \in \mathcal{F}$, G is H-free.
- **Theorem.** $\forall t \geqslant 2$, $ex(n, \{C_3, C_4, \cdots, C_{2t}\}) \leqslant n^{1+\frac{1}{t}} + n$
- **Lemma.** Every graph with at least dn edges has a subgraph whose minimum degree is at least d.
- **Proof of Lemma.** Delete vertices whose degrees are less than d. The subgraph left is not empty because the number of deleted edges is less than dn. ∎
- **Proof of Theorem.** Suppose there is a $\{C_3, C_4, \cdots, C_{2t}\}$-free n-vertices graph G with more than $n^{1+\frac{1}{t}} + n$ edges. By lemma, G has a subgraph G' with minimum degree $S(G') \geqslant n^{\frac{1}{t}} + 1$. Consider G' and the BFS-tree(Breadth-First Search).
Fix $v \in V(G')$, define $L_0 = v$, $L_1 = N(v)$
$L_{i+1} = \{w \in V(G') - \bigcup_{j=0}^{i} L_j : \exists u \in L_i, uw \in E(G')\} = N(L_i) - \bigcup_{j=0}^{i} L_j$
And G' is $\{C_3, C_4, \cdots, C_{2t}\}$-free, so $|L_i| \geqslant n^{\frac{1}{t}} |L_{i-1}|$, $|L_i| \geqslant n^{\frac{i}{t}}$, $|L_t| \geqslant n$, a contradiction.
This shows that $ex(n, \{C_3, C_4, \cdots, C_{2t}\}) \leqslant n^{1+\frac{1}{t}} + n$. ∎

$K_{t,t}$-free

- Recall: $ex(n, K_{s,t}) \leqslant Cn^{2-\frac{1}{t}}$ for $s \geqslant t \geqslant 2$
- **Conjecture.** $\forall t \geqslant 2$, $ex(n, K_{t,t}) = \Theta(n^{2-\frac{1}{t}})$
open for $t \geqslant 4$, need a lower bound.
- **Theorem(Erdös).** $ex(n, K_{2,2}) = \Theta(n^{\frac{3}{2}})$
- **Proof.** We construct an n-vertex graph with NO $K_{2,2}$ as follows.
Let $p \geqslant 2$ be a prime, $V(G) = \{(x, y) : x, y \in \mathbb{Z}_p, x \neq y\}$, $n = p(p-1)$. For (x, y), $(a, b) \in V(G)$, define $(x, y) \overset{G}{\backsim} (a, b)$ iff $xa + yb \equiv 1 \pmod p$. And $\forall v \in V(G)$, $d(v) = p$, we call G is p-regular.
Next we verify G is $K_{2,2}$-free. Suppose NOT, say $\exists (a1, b1), (a2, b2) \in V(G)$, s.t.
$$\begin{cases} a_1x + b_1y \equiv 1 \mod p \\ a_2x + b_2y \equiv 1 \mod p \end{cases}$$
has 2 solutions, a contradiction. ∎

- **Theorem(Brown).** $ex(n, K_{3,3}) = \Theta(n^{\frac{5}{3}})$

- **Proof.** Let $p \geqslant 2$ be a prime, define G by $V(G) = \{(x, y, z) : x, y, z \in \mathbb{Z}_p, x \neq y \neq z\}$, $n = p(p-1)(p-2)$. For (x, y, z), $(a, b, c) \in V(G)$, $(x, y, z) \overset{G}{\backsim} (a, b, c)$ iff $(x-a)^2 + (y-b)^2 + (z-c)^2 \equiv 1 \pmod p$. G is p-regular with $d = \Theta(p^2)$, $e(G) = \frac{1}{2}nd = \Theta(n^{\frac{5}{3}})$
Verify G has NO $K_{3,3}$. Suppose NOT, say $\exists(a1, b1, c1), (a2, b2, c2) \in V(G)$, s.t.

$$\begin{cases} (x-a_1)^2 + (y-b_1)^2 + (z-c_1)^2 \equiv 1 \mod p \\ (x-a_2)^2 + (y-b_2)^2 + (z-c_2)^2 \equiv 1 \mod p \\ (x-a_3)^2 + (y-b_3)^2 + (z-c_3)^2 \equiv 1 \mod p \end{cases}$$

has 3 solutions, a contradiction.(Consider in $\mathbb{R}^3$, since 3 spheres in general position intersect in at most 2 points.) ∎

<u>1-Distance Problems</u>

- Recall: $ex(n, K_{s,t}) = O(n^{1-\frac{1}{t}})$.
In particular, $ex(n, K_{2,3}) = O(n^{\frac{3}{2}})$

- **Problem 1.** In plane, given n vertices in general position, how many pairs of points have distance 1?

- Define a graph G on these n vertices, $a \overset{G}{\backsim} b$ iff the distance between a and b is 1.
Fact. G is $K_{2,3}$-free. Because two circle at a and b with radius 1 intersect in at most 2 points.

- **Theorem 1.** There are at most $O(n^{\frac{3}{2}})$ pairs at distance 1.

- **Proof.** Define G as above, so the number of pairs at distance 1 equals to $e(G)$, where G is $K_{2,3}$-free. ∎

- **Erdös.** Construct n points in plane with $n^{1+\frac{1}{\log \log n}}$ pairs at distance 1.

- **Open Problem.** Find an example of n points with n^{1+c} pairs at distance 1 for some $c > 0$.

- **Problem 2.** How many points in $\mathbb{R}^n$ s.t. the distance between any 2 points is 1.

- **Theorem 2.** In $\mathbb{R}^n$, there are at most $n+1$ points such that the distance between any 2 points is 1.

- **Proof.** Consider $\mathbb{R}^n$, say $\exists m$ points with distance 1 to each other. We may assume that one of the m points is $(0, 0, \cdots, 0)$, then the other $m-1$ points can be viewed as vectors, say $\boldsymbol{v_1}, \boldsymbol{v_2}, \cdots, \boldsymbol{v_{m-1}}$, and

$$\forall i, < \boldsymbol{v_i}, \boldsymbol{v_i} >= 1; \forall i \neq j, < \boldsymbol{v_i}, \boldsymbol{v_j} >= \frac{1}{2}$$

Let

$$A = \begin{pmatrix} \boldsymbol{v_1} \\ \boldsymbol{v_2} \\ \vdots \\ \boldsymbol{v_{m-1}} \end{pmatrix}_{(m-1)\times n} \qquad AA^T = \begin{pmatrix} 1 & 1/2 & \cdots & 1/2 \\ 1/2 & 1 & \cdots & 1/2 \\ \vdots & \vdots & \ddots & \vdots \\ 1/2 & 1/2 & \cdots & 1 \end{pmatrix}_{(m-1)\times(m-1)}$$

Since $\det(AA^T) \neq 0$, $m-1 = \text{rank}(AA^T) \leqslant \text{rank}(A) \leqslant n$, so $m \leqslant n+1$. ∎

2-Distance Problem

- How many points can we have in $\mathbb{R}^n$, such that their distance are one of 2 choices.
- **Example 1.** We can find $\binom{n}{2}$ such points.
  Consider 0-1 vectors whose 1-norm are exactly 2.
- **Example 2.** We can find $\binom{n+1}{2}$ such points in a n-dimension space.
  In $\mathbb{R}^{n+1}$, we can find $\binom{n+1}{2}$ such points according to Example 1. And those points are in the plane $\{\boldsymbol{x} = (x_1, x_2, \cdots, x_{n+1}) : \sum\limits_{i=1}^{n+1} x_i = 2\}$, which is a n-dimension space.
- **Theorem.** There are at most $\binom{n+2}{2} + n + 1$ points in $\mathbb{R}^n$ such that there distances are of 2 choices.
- **Proof.** Let $a^{(1)}, a^{(2)}, \cdots, a^{(m)} \in \mathbb{R}^n$ form a 2-distance set with distances d_1, d_2, write $a^{(i)} = (a_1^{(i)}, a_2^{(i)}, \cdots, a_n^{(i)})$. Define $f_i : \mathbb{R}^n \to \mathbb{R}$,

$$f_i(x) = (\left\|x - a^{(i)}\right\|^2 - d_1^2)(\left\|x - a^{(i)}\right\|^2 - d_2^2)$$

  we have $f_i(a^{(i)}) = d_1^2 d_2^2$ for $\forall i$, and $f_i(a^{(j)}) = 0$ for $\forall i \neq j$.
  Let $\lambda_1, \lambda_2, \cdots, \lambda_m \in \mathbb{R}$, and $\sum\limits_{i=1}^{m} \lambda_i f_i = 0$, that is $\sum\limits_{i=1}^{m} \lambda_i f_i(x) \equiv 0$ for $\forall x \in \mathbb{R}^n$.

$$\forall j, 0 = \sum_{i=1}^{m} \lambda_i f_i(a^{(j)}) = \lambda_j d_1^2 d_2^2 \Rightarrow \lambda_j = 0$$

  So $f_1, f_2, \cdots, f_m$ are linearly independent. They are all polynomials, and in the polynomial space which is spanned by

$$(\sum_{k=1}^{n} x_k^2)^2, (\sum_{k=1}^{n} x_k^2)x_i, x_i x_j, x_i, 1$$

  whose dimension is $\binom{n+2}{2} + n + 1$. So $m = \dim \text{span}\{f_1, f_2, \cdots, f_m\} \leqslant$ the dimension of the space above $= \binom{n+2}{2} + n + 1$. ∎

Odd-Town Problem

- **Problem.** Suppose we have a town with n people, they are assigned to form m clubs $A_1, A_2, \cdots, A_m$ such that
  (1) $|A_i|$ is odd.
  (2) $|A_i \bigcap A_j|$ is even for $\forall i \neq j$.
  How many clubs can we have?
- **Examples.** Here are some examples where $m = n$.
  (1) $A_i = \{i\}$
  (2) $A_i = [n] - \{i\}$, n is even.
  (3) $A_1 = [n] - \{1\}, A_2 = [n] - \{2\}$ and $A_i = \{1, 2, i\}$ for $i > 2$, n is even.

- **Theorem.** In the odd-town problem, $m \leqslant n$.
- **Proof.** For A_i, define 0-1 vector $\boldsymbol{v_i}$, $\boldsymbol{v_i}(j) = 1$ iff $j \in A_i$. We have
$<\boldsymbol{v_i}, \boldsymbol{v_i}> = |A_i|$ is odd.
$<\boldsymbol{v_i}, \boldsymbol{v_j}> = |A_i \bigcap A_j|$ is even for $\forall i \neq j$.
Claim: $\boldsymbol{v_1}, \boldsymbol{v_2}, \cdots, \boldsymbol{v_m}$ are linearly independent over $\mathbb{F}_2$.
Proof: Exercise.
Since $\boldsymbol{v_1}, \boldsymbol{v_2}, \cdots, \boldsymbol{v_m} \in \mathbb{F}_2^n$ are linearly independent, $m \leqslant n$.
Or we can let
$$A = \begin{pmatrix} \boldsymbol{v_1} \\ \boldsymbol{v_2} \\ \vdots \\ \boldsymbol{v_m} \end{pmatrix}_{m \times n} \qquad AA^T = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}_{m \times m} \qquad \text{over } \mathbb{F}_2$$
so $m = \text{rank}(AA^T) \leqslant \text{rank}A \leqslant n$. ∎

Even-Town Problem

- **Problem.** Suppose we have distinct $A_1, A_2, \cdots, A_m \subseteq [n]$, satisfying
(1) $|A_i|$ is even.
(2) $|A_i \bigcap A_j|$ is even for $\forall i \neq j$.
How much can m be?
- **Example.** Pair the n elements into $n/2$ pairs, say $S = \{(1,2), (3,4), \cdots, (n-1, n)\} = \{P_1, P_2, \cdots, P_{n/2}\}$, there are $2^{\frac{n}{2}}$ subsets of S. That is, $m = 2^{\frac{n}{2}}$.
- **Theorem.** In the even-town problem, $m \leqslant 2^{\frac{n}{2}}$.
- **Proof.** Define $\boldsymbol{v_i}$ as before, and let
$$A = \begin{pmatrix} \boldsymbol{v_1} \\ \boldsymbol{v_2} \\ \vdots \\ \boldsymbol{v_m} \end{pmatrix}_{m \times n}$$
rank$A$ =dim span$\{\boldsymbol{v_i}\}$, and rank$A$+dim Ker$A = n$(number of columns of A).
Case1: rank$A \leqslant \frac{n}{2}$
This shows that dim span$\{\boldsymbol{v_1}, \boldsymbol{v_2}, \cdots, \boldsymbol{v_m}\} \leqslant \frac{n}{2}$, so span$\{\boldsymbol{v_1}, \boldsymbol{v_2}, \cdots, \boldsymbol{v_m}\} \subseteq \mathbb{F}_2^{n/2}$, $m \leqslant 2^{\frac{n}{2}}$.
Case2: rank$A \geqslant \frac{n}{2} + 1$
This implies that we have at least $\frac{n}{2}+1$ linearly independent vectors, say $\boldsymbol{u_1}, \boldsymbol{u_2}, \cdots, \boldsymbol{u_{\frac{n}{2}+1}}$, consider
$$A\boldsymbol{u_i}^T = \begin{pmatrix} <\boldsymbol{v_1}, \boldsymbol{u_i}> \\ <\boldsymbol{v_2}, \boldsymbol{u_i}> \\ \vdots \\ <\boldsymbol{v_m}, \boldsymbol{u_i}> \end{pmatrix} = \begin{pmatrix} \text{even} \\ \text{even} \\ \vdots \\ \text{even} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \text{over } \mathbb{F}_2$$
so $\{\boldsymbol{u_1}, \boldsymbol{u_2}, \cdots, \boldsymbol{u_{\frac{n}{2}+1}}\} \subseteq \text{Ker}A$, dim Ker$A \geqslant \frac{n}{2}+1$. But it contradicts to rank$A$+dim Ker$A = n$. ∎