Combinatorial Networks
Week 7, Thursday, April 30

Matchings in Bipartite graphs: the Hall's Theorem

- **Definition.** A graph $G = (V, E)$ is called *bipartite*, if its vertices can be partitioned into two sets V_1 and V_2 such that any edge joints one in V_1 with another in V_2; equivalently, this means that there is no edge inside each V_i for $i = 1, 2$.

  We then call (V_1, V_2) as a *bipartition* of G. And it is well-known that graph G is bipartite if and only if every cycle in G has even length.

- **Definition.** A graph $G = (V, E)$ is *d-regular*, if the degree of any vertex in G is equal to d.

- **Definition.** A *matching* X in a graph $G = (V, E)$ is a collection of edges in E such that any two edges $e, f \in X$ do not share common vertices as their ends (and we will say e and f are independent).

  A matching X also determines a subgraph of graph G, which is 1-regular.

  We see that $|X| \le |V|/2$ for any matching X in graph $G = (V, E)$.

- A matching X of graph $G = (V, E)$ is *perfect*, if $|X| = |V|/2$.

- Given a matching X of graph G, a vertex u is called X*-matched*, if there exists some edge e of X such that e is incident to vertex u.

- Given a graph $G = (V, E)$ and a subset $A \subset V$ of vertices, a matching X is called an A*-perfect* matching, if $|X| = |A|$ and any vertex $a \in A$ is X-matched.

- For bipartite graph G with the bipartition (A, B) (assuming that $|A| \le |B|$), the largest possible matching is an A-perfect matching, which may not exits.

  We will study *the necessary and sufficient condition* for the existence of an A-perfect matching for bipartite graphs.

- **Definition.** For graph $G = (V, E)$ and subset $S \subset V$, the neighborhood of S in graph is defined as

  $$N_G(S) := \{v \in V - S : \text{there exists } u \in S \text{ such that } u \sim_G v\}.$$

  When there is no confusion, we also write $N_G(S)$ as $N(S)$.

- It follows by its definition that if G is bipartite with bipartition (A, B), then for any $S \subset A$, we have $N(S) \subset B$.

- The following classic theorem about bipartite graph G tells us when the largest possible matching, an A-perfect matching, exists in G.

  Hall's Theorem. Let $G = (V, E)$ be a bipartite graph with bipartition (A, B). Then, G has an A-perfect matching if and only if G satisfies the following so-called *Hall's condition* or *Marriage condition*:

  $$|N(S)| \ge |S|, \text{ for any subset } S \subset A.$$

- One direction of the proofs is easy: if G has an A-perfect matching X, then G satisfies Hall's condition.

  To see this, let $A = \{a_1, ..., a_k\}$, then we may assume that the A-perfect matching X is

  $$X = \{(a_i, b_i) : i = 1, ..., k\},$$

  where $a_i \in A, b_i \in B$. For any subset $S := \{a_{i_1}, ..., a_{i_s}\} \subset A$, we have $\{b_{i_1}, ..., b_{i_s}\} \subset N(S)$, implying that $|N(S)| \geq s = |S|$. So such G must satisfy the Hall's condition.

- We prove the another direction by induction on the size of A: if G is bipartite with bipartition (A, B) and for any $S \subset A$, $|N(S)| \geq |S|$, then G has an A-perfect matching.

  The basic case here is trivial: consider $|A| = 1$. We then make our inductive hypothesis saying that the desired statement holds for any G' with bipartition (A', B'), where $|A'| \leq |A|$. Here, A is from a bipartition (A, B) of G, which we are considering.

  We will divide the remaining proof into two cases.

  Case 1: for any subset $S \subset A$ (except $S = \emptyset$ and $S = V$), we have $|N(S)| \geq |S| + 1$.

  We pick any edge (a, b) with $a \in A, b \in B$ and consider $G' = G - \{a, b\}$. Note G' is still a bipartite graph with parts $A' = A - \{a\}, B' = B - \{b\}$. We check that G' always satisfies the Hall's condition (why?). Therefore by induction on $|A'| < |A|$, G' has an A'-perfect matching X'. Now $X = X' \cup \{(a, b)\}$ gives us the A-perfect matching of G!

  Case 2: there exists some $S \subset A$ with $0 < |S| < |A|$ and $|N(S)| = |S|$.

  Let $T = N(S) \subset B$ with $|S| = |T|$. Consider the subgraph G_1 induced by the vertex set $S \cup T$ and the subgraph $G_2 = G - G_1$. First, we see that G_1 and G_2 are bipartite as well. Then we check that both of G_1 and G_2 satisfy the Hall's condition (the proof here is omitted but you really need to see why it is the case!), therefore by induction, G_1 has an S-perfect matching X_1 and G_2 has an $(A - S)$-perfect matching X_2. Then $X = X_1 \cup X_2$ gives us the A-perfect matching of G as we want!

  This finishes the proof of the Hall's theorem.

Mathcings in general graphs: alternating/augmenting paths

- Before we introduce alternating/augmenting paths, we see an application of the Hall's Theorem.

- **Corollary.** For any integer $d \geq 1$, any d-regular bipartite graph G (say with bipartition A, B) has a perfect matching.

  The proof contains two steps. The first step is to show the two parts A and B are of equal size by considering the total number of edges, which equals $\sum_{v \in A} d(v) = d|A|$ and aslo equals $\sum_{v \in B} d(v) = d|B|$.

  The second step is to show that G satisfies Hall's condition, therefore G has an A-perfect matching, which in this case is also a perfect matching. To see the Hall's condition for G, consider any $S \subset A$. Let E_1 be the set of edges incident to S and let E_2 be the set of edges incidents to $N(S)$. By definition, we should have $E_1 \subset E_2$. But we also have

  $$|E_1| = \sum_{v \in S} d(v) = d|S| \quad \text{and} \quad |E_2| = \sum_{v \in N(S)} d(v) = d|N(S)|,$$

which implies that $|N(S)| \geq |S|$.

- We turn to study the matchings for general graphs (not necessary bipartite graphs now).

  Definition. Given a matching X of graph $G = (V, E)$,

  - a path $P = v_1 - v_2 - v_3 - \ldots - v_k$ in G is an X-*alternating path*, if the edges in P alternates between edges in X and edges not in X;
  - an X-alternating path $P = v_1 - v_2 - v_3 - \ldots - v_k$ is an X-*augmenting path*, if v_1, v_k are not X-matched.

- **Remark.** The intuition for the X-augmenting path is: if one can find such path P, then we can find a larger matching X' from X, by deleting all edges of P in X and adding all edges of P not in X!

- A graph $G = (V, E)$ is *connected* if for any two vertices u, v, there exists a path of G from u to v.

- A *component* of a graph $G = (V, E)$ is a *maximal* connected subgraph of G.

- We show the coming lemma first before raising our main theorem about augmenting path.

  Lemma. For any graph H, if degree of any vertex is at most 2, then any component of H is either an isolated vertex, or a path or a cycle. Moreover, each vertex of degree 1 must be an endpoint of some path in H.

- Sketch proof of Lemma. By induction on number of vertices. Base case is trivial. Now pick a vertex v with **minimum degree** in H. There are three cases.

  If $d(v) = 0$, then v is an isolated vertex; by induction on $H - v$, it is easy to see the statement holds for H.

  If $d(v) = 1$, then let u be the unique neighbor of v in H. The degree of vertex u in $H - v$ is either 1 or 0. By induction on $H - v$, the vertex u is either an isolated vertex of $H - v$ or is an endpoint of a path of $H - v$; in the later case, adding back edge (u, v), now v is an endpoint of a path in H!

  If $d(v) = 2$, then all vertices have degree 2. In this case, all vertices are of degree 2 in H, as $d(v) = 2$ is also the minimum degree of H. Then all vertices (except the two neighbors of v of degree 1) are of degree 2 in $H - v$. By induction on $H - v$, the two neighbors of v must be the two endpoints of a path P in $H - v$. Adding v back, then P becomes a cycle of H containing v. ∎

- The coming theorem tells us a way to determine the maximum matching for general graphs.

  Theorem. Let X be any matching in graph $G = (V, E)$. Then, X is a matching of G with maximum size if and only if there exist NO X-augmenting path in G.

- We prove one direction of first: if X is a matching of G with maximum size, then there is NO X-augmenting path.

  To see this, suppose for a contradiction, that there is an X-augmenting path P in G. We will also view P as the set of edges which are from path P. By the previous Remark, the obtained X' in fact is the symmetric difference $P \triangle X := P \cup X - P \cap X$, which is also a matching of G; moreover, we know P has one more edges not in X than edges in X (as P is X-augmenting), so we get $|X'| = |X| + 1$ (think why this is true), so X' turns out to be a matching with more edges than X, which is a contradiction to the assumption that X is maximum!

- For the another direction, we want to show that if there is no X-augmenting path, then X is maximum.

  Suppose for a contradiction that X is not maximum. Then, there is some matching X' with $|X'| > |X|$. Consider the subgraph $H := (V, X' \triangle X)$, where again $X' \triangle X$ is defined to be the symmetric difference between edge sets X' and X, that is $X' \cup X - X' \cap X$.

  Fact 1. H has more edges of X' than X, as its edge set $X' \triangle X$ is obtained from $X' \cup X$ by deleting the intersection $X' \cap X$.

  Fact 2. The degree of any vertex in H is at most 2. This is because all edges in H are from either X or X'; but every vertex can only have at most 1 edge from a matching.

  Therefore, by the lemma we proved above, any one of the components $D_1, D_2, ..., D_t$ in H is either an isolated vertex, or a path or a cycle. We now consider an arbitrary component D_i and want to compare the number of edges in D_i from X with the number of edges from X'.

  Case 0: when component D_i is an isolated vertex. There is no edge in D_i.

  Case 1: when component D_i is a cycle. Because the edges of cycle in H have to alternate between edges of X and edges of X', such cycle must be a cycle of even length with the equal number of edges from X and from X'.

  Case 2: when component D_i is a path. Then there are three types of paths. Note that the edges of path also have to alternate between edges of X and edges of X'.

  - Type I is a path with both of the initial edge and the last edge from X. Then path D_i has more edges of X than X'.
  - Type II is a path with one of initial edge and last edge from X and another from X'. Then path D_i has equal number of edges from X and from X'.
  - Type III is a path with both of the initial edge and the last edge from X'. Then path D_i has more edges of X' than X!

  By Fact 1, we know H (and therefore all $D_1, D_2, ..., D_t$) has more edges of X' than X. Notice that only the type III path will have more edges of X' than X; all other types have edges of X' which are no more than X! Therefore, there must be a component D_i of type III occurring! Such path D_i must be an X-augmenting path, which is a contradiction as the condition assumes no X-augmenting path. We finish the proof of theorem. ∎