Combinatorial Networks
Week 8, May 6-7

## Matching

- **Definition.** For $G$, a subset $U \subset V$ is called a vertex cover(or VC), if every edge in $G$ is incident to a vertex in $U$.

  $\Leftrightarrow U^c$ is an independent set in $G$.

- **Duality Theory.** any $|M| \leq$ any $|VC|$

  $\Leftrightarrow max|M| \leq min|VC|$

- **Theorem1(Konig, 1931).** For bipartite $G$, $max|M| = min|VC|$.

- **Proof.** Let $M$ be the maximum matching in $G$.

  A $M-$alternating path $P$ is $''$good$''$, if one of the ends in $P$ is in $A$ and not $M-$matched.

  We define a subset $U$, such that for any edge $ab \in M$, we will place exactly one of $ab$ in $U$.

  $$\begin{cases} b \in U, & \text{if } \exists \text{ ''good'' } M\text{-alternating path having } b \text{ as an end} \\ a \in U, & \text{otherwise} \end{cases}$$

  It satisfies for $U$ to be VC.

  Suppose not, $\exists ab \in E(G)$, $s.t. a \notin U, b \notin U$, which implies $ab \notin M$.

  **Claim1.** $b' \in B, s.t. ab' \in M$.

  **Proof.** Suppose not, then $a$ id not $M-$matched.

  As $M$ is $max$, $b$ must be $M-$matched.

  By definition, $ab$ is a $''$good$''$ $M-$alternating path.(For $b \in B, b \in U$, iff $b$ is $M-$matched and $\exists$ $''$good$''$ $M-$alternating path having $b$ as an end) (⁂)

  Hence, $b \in U$, a contradiction!

  **Claim2.** $\exists$ $''$good$''$ $M-$alternating path having $b$ as an end point.

  **Proof.** By the definition of $U$, since $a \notin U$, we have $b' \in U$.

  Thus, there is a $''$good$''$ $M-$alternating path $P'$, having $b'$ as its end.

  Let
  $$P = \begin{cases} P'b, & b \in P' \\ P'b'ab, & b \notin P' \end{cases}$$

  Thus, $P$ is a $''$good$''$ $M-$alternating path having $b$ as an end.

  If $b$ is not $M-$matched, then $P$ is a $M-$augumenting path, which is contrast with Berge's Theorem(as $M$ is $max$).

  Thus, $b$ isn't $M-$matched.

  By (⁂), we know $b \in U$. Contradiction again.

- **Theorem2.** For bipartite $G$ with $m$ edges, let $M$ be a matching. There is an $O(m)$ time algorithm for finding a $M-$augumenting path(if it exists).

- **Corollary1.** For bipartite $G$, $\exists O(nm)$ time algorithm for finding a maximum matching.

- **Proof.** Apply theorem2 by at most $\frac{n}{2}$ times.

- **Proof of theorem2.** Define a digraph as follows:

  (1) direct the edges in $M$ from $B$ to $A$, and other edges from $A$ to $B$.

  (2) add new vertex $x$ and arcs from $x$ to all unmatched vertices in $A$.

  We will take a $BSF-$tree $T$ with root $x$.It is enough to see if there is an unmatched vertex $b \in B$ in $T$.

  $--$If $\exists$ such $b$, then $\exists$ a directed path from unmatched vertex $a \in A$ to $b$, which is an $M-$augumenting path.

  $--$otherwise, no such $b$, then, there is no $M-$augumenting path.

- **Corollary2.** Given a maximum matching in bipartite $G$, we can find the smallest VC in $O(m)$ time.

  **Proof.** By the proof of theorem2 and the definition of $U$ in Theorem1.

- **Corollary3.** For bipartite $G$, $\exists O(nm)$ time algorithm for finding a minimum VC.

  **Proof.** Combine Corollary1 and Corollary2.

  **Hopcroft-Karp theorem**

- **Theorem3(Hopcroft-Karp).** For bipartite $G$, there is an $O(m\sqrt{n})$ time algorithm for finding a maximum matching in $G$.

- **Lemma1.** For general graph, let $M$ be a matching and $P$ be a $M-$augumenting path with the least length. Let $M' = M \triangle P$. Then any $M'-$augumenting path $P'$ satisfies that $|P'| \geq |P| + 2|P \cap P'|$.

  **Proof.** If $P \cap P' = \varnothing$, i.e. $P'$ shares no edges of $P$. Then $P'$ is also an $M-$augumenting path.Since $P$ is the shortest one, we have $|P'| \geq |P|$, done!

  If $|P \cap P'| \geq 1$

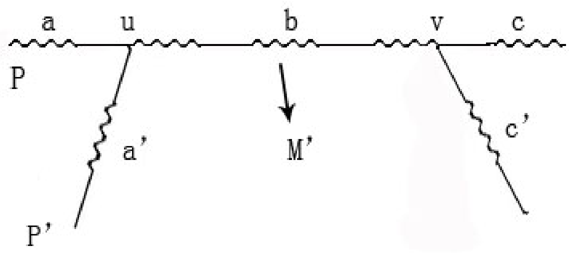


Figure 1:

From the figure, since $P$ is the shortest $M-$augumenting path, $c + c' \geq a + b + c$

Similarly, $a + a' \geq a + b + c$

Thus, $c' \geq a + b, a' \geq b + c$

$|P'| \geq a' + b + c' \geq (b + c) + b + (a + b) = (a + b + c) + 2b = |P| + 2b \geq |P| + 2|P \cap P'|$

- **Lemma2.** Let $M$ be a matching in bipartite $G$, then in time $O(m)$, we can find a maximal collection of vertex-disjoint $M-$augumenting paths of the shortest length.

  **Proof.** Similar to the previous proof.

  $--$Find the first layer of the $BFS-$tree, in which there is an unmatched vertex in $B$. Then pick such a vertex $b$.

  $--$Back tracking to get a directed path $P$ from $x$ to $b$, which is of the shortest length.

  $--$Delete all vertices of $P$ in the $BFS-$tree.

  $--$Repeat.

  Thus, obtain a maximal collection of $M-$augumenting paths of shortest length.

- **H.K algorithm.**

  Let $M = \varnothing$

  While there are $M-$augumenting path of length $k$

  $--$Let $k$ be the length of the shortest $M-$augumenting path

  $--$Find a maximal collection of, say $P_1, P_2, \cdots, P_t$ of vertex-disjoint $M-$augumenting path of length $k$

  $--$Let $M = M \vartriangle P_1 \vartriangle P_2 \vartriangle \cdots \vartriangle P_t$

- **Proof of Hopcroft-Karp theorem.** By lemma2, we can implement each iteration in $O(m)$ time

  Thus, it suffices to show that the $HK$ algorithm stops in $\leq 2\sqrt{n}$ iterations.

  **Claim.** In each iteration, the value of $k$ is increasing.

  Suppose claim holds. Then by the corollary, after $\sqrt{n}$ iterations, $|M^*| \leq |M| + \sqrt{n}$. Therefore, after $\sqrt{n}$ more iterations, this will stop.

  **Proof of claim.** Let $P_1, P_2, \cdots, P_t$ be the max collection of $M-$augumenting path of length $k$.

  Let $M' = M \vartriangle P_1 \vartriangle \cdots \vartriangle P_t$, $P'$ be any $M'-$augumenting path.

  We want to show $|P'| \geq k + 1$.

  **1)** $P'$ is edge-disjoint with $P_1, P_2, \cdots, P_t$

  Claim: $P'$ is vertex-disjoint with $P_1, P_2, \cdots, P_t$.

  Proof: Since $P'$ is edge-disjoint with $P_1, P_2, \cdots, P_t$, $|P'|$ is $M-$augumenting path.

  Assume $P'$ and $P_t$. has a common vertex $a$.

  (A)If $a$ is the middle point of $P'$.

  Then $a$ is $M-$matched. Thus, there is a common edge in $P'$ and $P_t$

  (B)If $a$ is the end point of $P'$.

Then $a$ is $M-$unmatched. Since, $M' = M \vartriangle P_1 \vartriangle \cdots \vartriangle P_t$, then $a$ is $M'-$matched. Thus, $P'$ is not $M'-$augumenting path. Contradiction!

**2)** $\exists P_t$, *s.t.* $P'$ and $P_t$ share a common edge.

Apply lemma1 to $P_t, M \vartriangle P_1 \vartriangle \cdots \vartriangle P_t - 1$ and $P'$

claim, $P_t$ is also the $M \vartriangle P_1 \vartriangle \cdots \vartriangle P_t - 1-$augumenting path of shortest length.

On the other hand, $P'$ is $(M \vartriangle P_1 \vartriangle \cdots \vartriangle P_t - 1) \vartriangle P_t-$augumenting path by lemma1.

Thus, $|P'| \geq |P_t| + 2|P' \cap P_t| \geq k + 2$