Linear cycles of consecutive lengths

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Abstract

A well-known result of Verstraëte [43] shows that for each integer $k \geq 2$ every graph G with average degree at least $8k$ contains cycles of k consecutive even lengths, the shortest of which is at most twice the radius of G. We establish two extensions of Verstraëte's result for linear cycles in linear r-uniform hypergraphs.

We show that for any fixed integers $r \geq 3$ and $k \geq 2$, there exist constants $c_1 = c_1(r)$ and $c_2 = c_2(r)$, such that every linear r-uniform hypergraph G with average degree $d(G) \geq c_1 k$ contains linear cycles of k consecutive even lengths, the shortest of which is at most $2\lceil \frac{\log n}{\log(d(G)/k) - c_2} \rceil$. In particular, as an immediate corollary, we retrieve the current best known upper bound on the linear Turán number of C_{2k}^r with improved coefficients.

Furthermore, we show that for any fixed integers $r \geq 3$ and $k \geq 2$, there exist constants $c_3 = c_3(r)$ and $c_4 = c_4(r)$ such that every n-vertex linear r-uniform graph with average degree $d(G) \geq c_3 k$, contains linear cycles of k consecutive lengths, the shortest of which has length at most $6\lceil \frac{\log n}{\log(d(G)/k) - c_4} \rceil + 6$. Both the degree condition and the shortest length among the cycles guaranteed are best possible up to a constant factor.

1 Introduction

For $r \geq 3$, an r-uniform hypergraph (henceforth, r-graph) is *linear* if any two edges share at most one vertex. An r-uniform *linear cycle* of length k, denoted by C_k^r, is a linear r-graph consisting of k edges $e_1, e_2, ..., e_k$ on $(r-1)k$ vertices such that $|e_i \cap e_j| = 1$ if $j = i \pm 1$ (indices taken modulo k) and $|e_i \cap e_j| = 0$ otherwise. For $r = 2$, linear r-graphs are just the usual graphs, and so are the linear cycles. Motivated by the known results for graphs, we study sufficient conditions for the existence of linear cycles of given lengths in linear r-graphs for $r \geq 3$. Our results apply to linear r-graphs of a broad edge density, covering both sparse and dense graphs.

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1.1 History

The line of research about the distribution of cycle lengths in graphs was initiated by Burr and Erdős (see [9]) who conjectured that for every odd number k, there is a constant c_k such that for every natural number m, every graph of average degree at least c_k contains a cycle of length m modulo k. This conjecture was confirmed in this full generality by Bollobás [2] for $c_k = 2((k+1)^k - 1)/k$, although earlier partial results were obtained by Erdős and Burr [9] and Robertson [9]. The constant c_k was improved to $8k$ by Verstraëte [43]. Thomassen [40, 41] strengthened the result of Bollobás by proving that for every k (not necessarily odd), every graph with minimum degree at least $4k(k+1)$ contains cycles of all even lengths modulo k.

On a similar note, Bondy and Vince [4] proved a conjecture of Erdős in a strong form showing that any graph with minimum degree at least three contains two cycles whose lengths differ by one or two. Since then there has been extensive research (such as [24, 15, 38, 33, 32]) on the general problem of finding k cycles of consecutive (even or odd) lengths under minimum degree or average degree conditions in graphs. Very recently, the optimal minimum degree condition assuring the existence of such k cycles was announced in [20].

The problem of finding consecutive length cycles in r-graphs is related to another classical problem in extremal graph theory, namely Turán numbers for cycles in graphs and hypergraphs. For $r \geq 2$, the *Turán number* $\mathrm{ex}(n, \mathcal{F})$ of a family $\mathcal{F}$ of r-graphs is the maximum number of edges in an n-vertex r-graph which does not contain any member of $\mathcal{F}$ as its subgraph. If $\mathcal{F}$ consists of a single graph F, we write $\mathrm{ex}(n, F)$ for $\mathrm{ex}(n, \{F\})$. A well-known result of Erdős (unpublished) and independently of Bondy and Simonovits [3] states that for any integer $k \geq 2$, there exists some absolute constant $c > 0$ such that $\mathrm{ex}(n, C_{2k}) \leq ckn^{1+1/k}$. The value of c was further improved by the results of Verstraëte [43] and Pikhurko [35], and the current best known upper bound is $\mathrm{ex}(n, C_{2k}) \leq 80\sqrt{k \log k}\, n^{1+1/k} + O(n)$, due to Bukh and Jiang [5]. Verstraëte's main result from [43] is as follows.

Theorem 1.1 (Verstraëte, [43]) *Let $k \geq 2$ be an integer and G a bipartite graph of average degree at least $4k$ and girth g. Then there exist cycles of $(g/2-1)k$ consecutive even lengths in G, the shortest of which has length at most twice the radius of G.*

In Theorem 1.1, in addition to finding k cycles of consecutive even lengths we also see an upper bound on the shortest length among these cycles. Thus it immediately yields $\mathrm{ex}(n, C_{2k}) \leq 8kn^{1+1/k}$, which improves on the coefficients in the theorems of Erdős and of Bondy-Simonovits. Notice that Verstraëte's theorem is applicable to both sparse and dense host graphs while arguments establishing bounds on $\mathrm{ex}(n, F)$ directly usually address relatively dense host graphs. For example, for $F = C_{2k}$, these would typically be graphs with average degree at least $\Omega(n^{1/k})$.

For hypergraphs, Verstraëte [44] conjectured that for $r \geq 3$ any r-graph with average degree $\Omega(k^{r-1})$ contains Berge cycles of k consecutive lengths where an r-uniform *Berge cycle* of length k is a hypergraph containing k vertices $v_1, ..., v_k$ and k distinct edges $e_1, ..., e_k$ such that $\{v_i, v_{i+1}\} \subseteq e_i$ for each i, where the indices are taken modulo k. Let $\mathcal{B}_k^r$ denote the family of r-uniform Berge cycles of length k. Results of [18, 21, 22, 23] showed that for all $k, r \geq 3$, $\mathrm{ex}(n, \mathcal{B}_k^r) \leq c_{k,r} \cdot n^{1+1/\lfloor k/2 \rfloor}$, where $c_{k,r} = O(k^r)$. Jiang and Ma in [25] confirmed Verstraëte's conjecture on the existence of Berge cycles of consecutive lengths, and just as in Theorem 1.1, they were able to control the length of the shortest cycle in the collection which implied an improved $c_{k,r}$ by an $\Omega(k)$ factor in the upper bound of $\mathrm{ex}(n, \mathcal{B}_k^r)$. As an intermediate step and a result of independent interest, they also proved the following result.

Theorem 1.2 (Jiang and Ma, [25]) *For all $r \geq 3$, any linear r-graph with average degree at least $7r(k+1)$ contains Berge cycles of k consecutive lengths.*

Theorem 1.2 suggests that the problem of finding Berge cycles of consecutive lengths in general r-graphs bears some resemblance to the graph case, but that is not the case for linear cycles. Indeed, the Turán number $\mathrm{ex}(n, C_k^r)$ of the linear cycle C_k^r was determined precisely for large n by Füredi and Jiang [17] for $r \geq 5$ and independently by Kostochka, Mubayi and Verstraëte [29] for $r \geq 3$. Asymptotically their results show that $\mathrm{ex}(n, C_k^r) \sim \lfloor \frac{k-1}{2} \rfloor \binom{n}{r-1}$.

However, if we study the emergence of linear cycles in linear host hypergraphs instead then the behavior of the Turán numbers of linear cycles bears much more resemblance to the graph case. To be more precise, let us define the *linear Turán number* $\mathrm{ex}_L(n, H)$ of a linear r-graph H to be the maximum number of edges in an n-vertex linear r-graph G that does not contain H as a subgraph. Quoting [44], the problem of determining the linear Turán number of a linear cycle "seems to be a more faithful generalization of the even cycle problem in graphs". Indeed, Collier-Cartaino, Graber, and Jiang [6] proved that for all integers $r, k \geq 2$ there exist positive constants $c(r, k)$ and $d(r, k)$ such that $\mathrm{ex}_L(n, C_{2k}^r) \leq c(r, k)n^{1+1/k}$ and $\mathrm{ex}_L(n, C_{2k+1}^r) \leq d(r, k)n^{1+1/k}$. For fixed r, the constants $c(r, k)$ and $d(r, k)$ established are exponential in k. As a corollary, one of the main results we prove, Theorem 1.3 implies that $c(r, k)$ can be taken linear in k, improving the results in [6]. Note that these results on linear Turán numbers of linear even cycles can be viewed as a generalization of the Bondy-Simonovits even cycle theorem, while the result on odd linear cycles demonstrates a phenomenon that is very different from the graph case. To this end, note that the study of $\mathrm{ex}_L(n, C_3^3)$ is equivalent to the famous $(6, 3)$-problem, which is to determine the maximum number of edges $f(n, 6, 3)$ in an n-vertex 3-graph such that no six vertices span three or more edges. Ruzsa and Szemrédi [36] showed that for some constant $c > 0$, $n^{2-c\sqrt{\log n}} < f(n, 6, 3) = o(n^2)$, where the upper bound uses the regularity lemma and the lower bound uses Behrend's construction [1] of dense subsets of $[n]$ not containing 3-term arithmetic progressions.

1.2 Our results

We establish two extensions of Theorem 1.1 for linear cycles in linear r-uniform hypergraphs. First, we give a generalization of Theorem 1.1 for even linear cycles in linear r-graphs along with a near optimal control on the shortest length of the even cycles obtained.

Theorem 1.3 *Let $r \geq 3$ and $k \geq 2$ be integers. There exist constants c_1, c_2 depending on r such that if G is an n-vertex linear r-graph with average degree $d(G) \geq c_1 k$ then G contains linear cycles of k consecutive even lengths, the shortest of which is at most $2\lceil \frac{\log n}{\log(d(G)/k) - c_2} \rceil$.*

Theorem 1.3 immediately implies an improved upper bound on the linear Turán number of linear even cycles which previously was $cn^{1+1/k}$ for some c exponential in k [6], for fixed r.

Corollary 1.4 *Let $r \geq 3$ and $k \geq 2$ be integers. There exists a constant c, depending only on r such that for all positive integers n, we have*

$$ex_L(n, C_{2k}^r) \leq ckn^{1+1/k}.$$

Our next main result shows that under analogous degree conditions as in Theorem 1.3, we can in fact ensure linear cycles of k consecutive lengths (even and odd both included), not just linear cycles of k consecutive even lengths. Furthermore, the length of the shortest cycle in the collection is within a constant factor of being optimal. Note that such a phenomenon can only exist in r-graphs with $r \geq 3$, as for graphs, one needs more than $n^2/4$ edges in an n-vertex graph just to ensure the existence of any odd cycle.

Theorem 1.5 *Let $r \geq 3$ and $k \geq 1$ be integers. There exist constants c_1, c_2 depending on r such that if G is an n-vertex linear r-graph with average degree $d(G) \geq c_1 k$ then G contains linear cycles of k consecutive lengths, the shortest of which is at most $6\lceil \frac{\log n}{\log(d(G)/k) - c_2} \rceil + 6$.*

When viewed as a result on the average degree needed to ensure cycles of consecutive lengths, Theorem 1.5 is a substantial strengthening of both Theorem 1.2 and Theorem 1.3. However, the control on the shortest length of a cycle in the collection is weaker than those in Theorem 1.3 and in [25] by roughly a factor of 3. As a result, while Theorem 1.3 yields $\mathrm{ex}_L(n, C^r_{2k}) = O(n^{1+1/k})$, Theorem 1.5 would only give us $\mathrm{ex}(n, C^r_{2k+1}) = O(n^{1+3/k})$, and hence it does not imply the bound on $\mathrm{ex}_L(n, C^r_{2k+1})$ given in [6].

Finally, note that the shortest lengths of linear cycles that we find in Theorem 1.3 and Theorem 1.5 are within a constant factor of being optimal, due to the following proposition which can be proved using a standard deletion argument. We delay its proof to the appendix.

Proposition 1.6 *Let $r \geq 2$ be an integer. For every real $\epsilon > 0$ there exists a positive integer n_0 such that for all integers $n \geq n_0$ and for each d satisfying $(2r)^{\frac{1}{\epsilon^2}} \leq d \leq n/4$, there exists an n-vertex linear r-graph with average degree at least d and containing no linear cycles of length at most $(1-\epsilon)\log_d n$.*

The rest of the paper is organized as follows. In Section 2, we introduce some notation. In Section 3, we prove Theorem 1.5. In Section 4, we prove Theorem 1.3, whose proof is more involved than that of Theorem 1.5 due to the tighter control on the shortest lengths of the cycles. In Section 5, we conclude with some remarks and problems for future study on related topics.

2 Notation

Let $r \geq 2$ be an integer. Given an r-graph G, we use $\delta(G)$ and $d(G)$ to denote the minimum degree and the average degree of G, respectively. Given a graph G and and a set S, an *edge-colouring* of G using subsets of S is a function $\chi : E(G) \to 2^S$. We say that χ is *strongly proper* if $V(G) \cap S = \emptyset$ and whenever e, f are two distinct edges in G that share an endpoint we have $\chi(e) \cap \chi(f) = \emptyset$. We say that χ *strongly rainbow* if $V(G) \cap S = \emptyset$ and whenever e, f are distinct edges of G we have $\chi(e) \cap \chi(f) = \emptyset$.

For $r \geq 2$, an r-graph G is *r-partite* if there exists a partition of $V(G)$ into r subsets $A_1, A_2 \ldots, A_r$ such that each edge of G contains exactly one vertex from each A_i; we call such $(A_1, \ldots, A_r)$ an *r-partition* of G. For any $1 \leq i \neq j \leq r$, we define the *(A_i, A_j)-projection* of G, denoted by $P_{A_i,A_j}(G)$ to be the graph with edge set $\{e \cap (A_i \cup A_j) | \, e \in E(G)\}$. It is easy to see that for linear r-partite r-graphs the following mapping $f : E(G) \to E(P_{A_i,A_j}(G))$ defined by $f(e) = e \cap (A_i \cup A_j)$ is bijective.

In this paper, logarithms are base 2 and $[k]$ denotes the set $\{1, 2, ..., k\}$ for all positive integers k.

3 Linear cycles of consecutive lengths

In this section, we prove Theorem 1.5. We prove this theorem first, since the proof is relatively short. A key notion needed for our proof is that of distance in linear hypergraphs. Given a linear r-graph G and two vertices x, y in G, we define the *distance* $d_G(x, y)$ to be the length of a shortest linear path between x and y. We drop the index G whenever the context is clear. For any vertex $x \in V(G)$, we define $S^G_0(x) = \{x\}$ and for all $i \geq 1$ define

$$S^G_i(x) = \{y \in V(G) : d_G(x, y) = i\}.$$

When G and/or x are clear, we will drop the super- and/or subscript.

We first prove some auxiliary lemmas that are used in the proof of Theorem 1.5. Our first lemma is folklore.

Lemma 3.1 *Let $r \geq 2$ be an integer and $d > 0$ a real. Every r-graph G of average degree d contains a subgraph of minimum degree at least d/r.*

Lemma 3.2 *Let $r \geq 3$ be an integer. Let G be a linear r-graph. Let d be a real satisfying $1 \leq d \leq \delta(G)/2$. Let $x \in V(G)$. Then there exist a positive integer $m \leq \lceil \frac{\log n}{\log(\delta(G)/d)} \rceil$ and a subgraph H of G satisfying*

(A1) H has average degree at least $d/4$, and

(A2) each edge of H contains at least one vertex in $S_m(x)$ and no $\bigcup_{j<m} S_j(x)$.

Proof. For each $i \geq 0$, let $S_i = S_i(x)$. By the definition of the S_i's, for each $e \in E(G)$, there exists $j \geq 0$ such that $e \subseteq S_j \cup S_{j+1}$. For each $i \geq 1$ let G_i be the subgraph of G induced by the edges that contain some vertex in S_i. Then $V(G_i) \subseteq S_{i-1} \cup S_i \cup S_{i+1}$. Let $t = \lceil \frac{\log n}{\log(\delta(G)/d)} \rceil$. First we show that for some $i \in [t]$, G_i has average degree at least $d/2$. Suppose for contradiction that for each $i \in [t]$, G_i has average less than $d/2$. Then for each $i \in [t]$, $e(G_i) \leq (d/2)|V(G_i)|/r \leq (d/2r)(|S_{i-1}| + |S_i| + |S_{i+1}|)$. On the other hand, by minimum degree condition we have $e(G_i) \geq \delta(G)|S_i|/r$. Combing the two inequalities, we get

$$|S_{i-1}| + |S_i| + |S_{i+1}| \geq \frac{2\delta(G)}{d}|S_i|. \tag{1}$$

Claim 3.3 *For each $i \in [t]$, we have $|S_i| > (\delta(G)/d)|S_{i-1}|$.*

Proof. The claim holds for $i = 1$ since $|S_1| \geq \delta(G)$ and $|S_0| = 1$. Let $1 \leq j < t$ and suppose the claim holds for $i = j$. We prove the claim for $i = j + 1$. By (1) and the induction hypothesis that $S_{j-1} \leq (d/\delta(G))|S_j|$, we have

$$\frac{d}{\delta(G)}|S_j| + |S_j| + |S_{j+1}| \geq \frac{2\delta(G)}{d}|S_j|.$$

Hence

$$|S_{j+1}| \geq (\frac{2\delta(G)}{d} - \frac{d}{\delta(G)} - 1)|S_j| > \frac{\delta(G)}{d}|S_j|,$$

where the last inequality uses $d \leq \delta(G)/2$. ■

By the claim, $|S_t| > (\frac{\log n}{\log(\delta(G)/d)})^t \geq n$, which is a contradiction. So there exists $i \in [t]$ such that G_i has average degree at least $d/2$. By our earlier discussion, each edge of G_i contains a vertex in S_i and lies inside $S_{i-1} \cup S_i \cup S_{i+1}$. If at least half of the edges of G_i contain some vertex in S_{i-1} then let H be the subgraph of G_i consisting of these edges and let $m = i - 1$. Otherwise, let H be the subgraph of G_i consisting of edges that do not contain vertices of S_{i-1} and let $m = i$. In either case, H and m satisfy (A1) and (A2). ■

Lemma 3.4 *Let $r \geq 3$. Let G be a linear r-graph. Let d be a real satisfying $1 \leq d \leq \delta(G)/2$. Let $x \in V(G)$. For each $v \in V(G)$, let P_v be a fixed shortest (x,v)-path in G and let $\mathcal{P} = \{P_v : v \in V(G)\}$. Then there exist a positive integer $m \leq \lceil \frac{\log n}{\log(\delta(G)/d)} \rceil$, $A \subseteq S_m(x)$ and a subgraph F of G such that the following hold:*

(P1) $\delta(F) \geq d/r2^{2r+1}$,

(P2) each edge of F contains exactly one vertex from A and no vertices from the set $\bigcup_{j<m} S_j(x)$,

(P3) for each $v \in V(F) \cap A$, P_v intersects $V(F)$ only in v.

Proof. By Lemma 3.2, there exist a subgraph H of G and a positive integer $m \leq \lceil \frac{\log n}{\log(\delta(G)/d)} \rceil$ satisfying properties (A1)-(A2). So, in particular, $d(H) \geq d(G)/4$. Now let $X \subseteq V_m$ be obtained by including each vertex of V_m independently with probability 1/2. We call an edge $f \in E(H)$ *good* if $|f \cap X| = 1$. For each such $f \in E(H)$ the probability of it being good is $|f \cap V_m|/2^r \geq 1/2^r$. So there exists a choice of X such that the subgraph of H formed by the good edges, call it H', satisfies $e(H') \geq e(H)/2^r$. Fix such a choice of X and the corresponding H'. For every edge $f \in E(H')$ let v_f be the unique vertex in $f \cap X$ and e_{v_f} be the edge in the path P_{v_f} which contains v_f.

Now, let Y be a random subset of X obtained by choosing each vertex of X independently with probability 1/2. For each edge $f \in E(H')$, we call f *nice* if $e_{v_f} \cap Y = \{v_f\}$. Given any $f \in E(H')$, the probability of f being nice is $(1/2)^{|e_{v_f} \cap X|} \geq (1/2)^{r-1}$ as $v_f \in e_{v_f} \cap X$ and $|e_{v_f} \cap X| \leq r-1$. So there exists a choice of Y such that the subgraph of H' formed by the nice edges, call it H'', satisfies

$$d(H'') \geq \frac{d(H')}{2^{r-1}} \geq \frac{d(H)}{2^{2r-1}} \geq \frac{d(G)}{2^{2r+1}}.$$

Fix such a choice of Y and H'', set $A := Y$. By Lemma 3.1 H'' has a subgraph F of minimum degree at least $d(H'')/r \geq d/r2^{2r+1}$. Now, A and F satsify (P1)-(P3). ∎

Lemma 3.5 *Let $r \geq 3$, $k \geq 1$ be integers. Let F be a linear r-graph and $A \subset V(F)$ be such that each edge of F contains exactly one vertex of A. If $\delta(F) \geq rk$ then F contains a linear path of length $k+2$ such that each vertex in $V(P) \cap A$ has degree one in P.*

Proof. Let P be a longest linear path in F with the property that vertices in $V(P) \cap A$ have degree one in P. Let e be an end edge of P. Since e has $r-1 \geq 2$ vertices of degree one in P and $|e \cap A| = 1$, there exists a vertex $v \in e \setminus A$ that has degree one in P. There at least $\delta(G) \geq rk$ edges of G containing v. Since G is linear, there are at most $|V(P)| - r + 1$ edges in G that contain v and another vertex on P. Suppose $|V(P)| - r + 1 < rk$. Then there is an edge f in G that contains v and no other vertex on P. But now $P \cup f$ is a longer path than P and each vertex in $V(P \cup f) \cap A$ has degree one in $P \cup f$, contradicting our choice of P. Hence $|V(P)| \geq rk + r - 1$, which implies $|P| \geq k+2$. ∎

Lemma 3.6 *Let $r \geq 3, k \geq 1$, $d = kr^2 2^{2r+2}$. Let F be a linear r-graph with $\delta(F) \geq 2d$ and x be any vertex in F. Then there exist edges e and f and some integer $t \leq \lceil \frac{\log n}{\log(\delta(F)/d)} \rceil$ such that for each $i \in [k]$, there is a path of length $t+2+i$ starting at x and having e and f as its last two edges.*

Proof. For each vertex v in F, let P_v be a shortest (x,v)-linear path in F. By Lemma 3.4 (with F playing the role of G) there exist a positive integer $t \le \lceil \frac{\log n}{\log(\delta(F)/d)} \rceil$, a subset $A \subseteq S_t(x)$ and a subgraph F' of F that satisfy (P1)-(P3). In particular, $\delta(F') \ge d/r2^{2r+2} = rk$. Applying Lemma 3.5 to F', we obtain a linear path P of length $k+2$ in F' such that each vertex in $V(P) \cap A$ has degree one in P. Suppose the edges of P are ordered as $e_1, \dots, e_k, e, f$. For each $i \in [k]$, let v_i be the unique vertex in $e_i \cap A$. For each $i \in [k]$ since P_{v_i} intersects $V(F)$ only in v_i, $P_{v_i} \cup \{e_i, \dots, e_k, e, f\}$ is a linear path of length $(k+2)-(i-1)+t$ that starts at x and ends with e, f. Since this holds for each $i = 1, \dots, k$, the claim follows. ∎

Now we are ready to prove Theorem 1.5.

Proof of Theorem 1.5: We will show the statement holds for $c_1 = 2^{4r+8}r^3, c_3 = 2^{4r+4}r^5, c_2 = \log(c_3)$. By Lemma 3.1 G contains a subgraph G' with $\delta(G') \ge d(G)/r$. Set $\delta = d(G)/r$. Then $\delta(G') \ge \delta$. Set

$$d' = kr^2 2^{2r+2}, \quad d = r^{3/2} 2^{2r+2} \sqrt{\delta k}.$$

Note that when k, r are fixed, d' is a constant, but $d = \Theta(\sqrt{d(G)})$. Our d is chosen to approximately optimize the upper bound we obtain later on the lengths of the cycles.

Let x_0 be any vertex in G'. By Lemma 3.4 there exist $m \le \lceil \frac{\log n}{\log(\delta/d)} \rceil$, a subset $A \subseteq S_m(x_0)$ and a subgraph F of G' such that

(P1) $\delta(F) \ge \frac{d}{r2^{2r+2}}$,

(P2) each edge of F contains exactly one vertex in A but no vertex in $\bigcup_{j<i} S_j(x_0)$, and

(P3) for each $v \in V(F) \cap A$, P_v intersects $V(F)$ only in v.

Now let x be any vertex in $V(F) \cap A$. Since $\delta(F) \ge d/r2^{2r+2} \ge 2d'$, by Lemma 3.6, there exist two edges e and f in F and some integer $t \le \lceil \frac{\log n}{\log(\delta(F)/d')} \rceil$ such that for each $i \in \{t+3, t+4, \dots, t+k+2\}$ there is a linear a path Q_i in F of length i which starts at x and has e and f as the last two edges.

Let y be the unique vertex in $A \cap f$. By (P3), P_x and P_y intersect $V(F)$ only in x and y, respectively. Therefore, $P_x \cup P_y$ must contain a linear (x,y)-path of length $q \le 2m$ that intersects $V(F)$ only in x and y. Let us denote this subpath by P_{xy}.

If $y \notin e \cap f$, then $P_{xy} \cup Q_1, \dots, P_{xy} \cup Q_k$ are linear cycles of lengths $q+t+3, \dots, q+t+k+2$, respectively. If $y \in e \cap f$ then $P_{xy} \cup (Q_1 \setminus f), \dots, P_{xy} \cup \{Q_k \setminus f)$ are linear cycles of lengths $q+t+2, \dots, q+t+k+1$, respectively. In either case we find linear cycles of k consecutive lengths, the shortest of which has length at most

$$q+t+3 \le 2m+t+3 \quad \le \quad 2\left\lceil \frac{\log n}{\log(\delta/d)} \right\rceil + \left\lceil \frac{\log n}{\log(\delta(F)/d')} \right\rceil + 3 \le 3\left\lceil \frac{\log n}{\log(\delta/d)} \right\rceil + 3,$$

where the last inequality holds since $\delta(F)/d' \ge \delta/d$. By our choice of d and c_3, we can check that $\delta/d \ge (d(G)/kc_3)^{1/2}$. Hence, the above upper bound is at most

$$3\left\lceil \frac{\log n}{(1/2)\log(d(G)/c_3 k)} \right\rceil + 3 \le 3\left\lceil \frac{2\log n}{\log(d(G)/k) - c_2} \right\rceil + 3 \le \frac{6\log n}{\log(d(G)/k) - c_2} + 6.$$

This completes the proof of Theorem 1.5. ∎

4 Sharper results for linear cycles of even consecutive lengths

For linear cycles of even consecutive lengths, we obtain much tighter control on the shortest length of a cycle in the collection, which as a byproduct also gives us an improvement on the current best known upper bound on the linear Turán number $\mathrm{ex}_L(n, C^r_{2k})$ of an r-uniform linear cycle of a given even length $2k$. The previous best known upper bound is $c_{r,k}n^{1+1/k}$, where $c_{r,k}$ is exponential in k for fixed r. For fixed r, we are now able to improve the bound on $c_{r,k}$ to a linear function of k.

4.1 A useful lemma on long paths with special features

One of the key ingredients of our proof of the main result in this section is Lemma 4.2. The lemma is about the existence of a long path with special features in a properly edge-colored graph with high average degree. It may be viewed a strengthening of two lemmas used in [25] (Lemma 2.6 and Lemma 2.7). We start with a preliminary lemma.

Lemma 4.1 *Let G be a connected n-vertex graph with average degree at least $2d$. Then there exists a linear ordering σ of $V(G)$ as $x_1 < x_2 < \cdots < x_n$ and some $0 \le m < n$ such that for each $1 \le i \le m$ $|N_G(x_i) \cap \{x_{i+1}, \ldots, x_n\}| < d$ and that the subgraph F of G induced by $\{x_{m+1}, \ldots, x_n\}$ has minimum degree at least d.*

Proof. As long as G contains a vertex whose degree in the remaining subgraph is less than d we delete it from G. We continue until no such vertex exists. Let F denote the remaining subgraph. Suppose this terminates after m steps. Then we have deleted at most $dm \le d(n-1) < e(G)$ edges. Hence F is nonempty. Let $x_1 < x_2 < \cdots < x_m$ be the vertices deleted in that order. Let $x_{m+1} < \ldots < x_n$ be an arbitrary linear ordering of the remaining vertices. Then the ordering $\sigma := x_1 < \ldots < x_n$ and F satisfy the requirements. ∎

The following lemma is written in terms of colourings of graphs, but in our applications the graph H will be some (A_i, A_j)-projection of an r-partite r-graph G where the colouring is obtained by colouring the edge $e \cap (A_i \cup A_j)$ in H by the $(r-2)$-set $e \setminus (A_i \cup A_j)$ for each $e \in E(G)$.

Lemma 4.2 *Let $r \ge 3$ and $\ell \ge 2$. Let H be a connected graph with minimum degree at least $4r\ell$. Let χ be a strongly proper edge-colouring of H using $(r-2)$-sets. Let E_1, E_2 be any partition of $E(H)$ into two nonempty sets such that $|E_1| \le |E_2|$. Then there exists a strongly rainbow path of length at least ℓ in H such that the first edge of P is in E_1 and all the other edges are in E_2.*

Proof. For $i = 1, 2$, let H_i be the subgraph of H induced by the edge set E_i. Note that $d(H_2) \ge 2r\ell$. Let L be a connected component of H_2 with $d(L) \ge 2r\ell$. By Lemma 4.1 (applied to L), there exist some integer $0 \le m < |V(L)|$ and a linear ordering $\sigma := x_1 < x_2 < \cdots < x_{|V(L)|}$ of $V(L)$ such that for each $1 \le i \le m$, $N_L(x_i) \cap \{x_{i+1}, \ldots x_{|V(L)|}\}| < r\ell$ and that the subgraph F of L induced by $\{x_{m+1}, \ldots, x_{|V(L)|}\}\}$ has minimum degree at least $r\ell$.

Let us call a strongly rainbow path P in H a *good path* if it has length at least one, its first edge is in E_1 and its other edges (if exist) are in E_2. To prove the lemma, we need to show that H has a good path of length ℓ.

Claim 4.3 *If H has a good path that ends with a vertex in F, then H has a good path of length ℓ.*

Proof. Among all good paths in H that end with a vertex in F, let $P = uv_1v_2 \ldots v_j$ be a longest one (subject to $uv_1 \in E_1$ and $v_j \in V(F)$). If $j \ge \ell$ then we are done. Hence we may assume that

$j \leq \ell - 1$. Since $\delta(F) \geq r\ell$, there are at least $r\ell$ edges of F incident to v_j. Among these edges, more than $\ell r - j > \ell(r-1)$ of them join v_j to a vertex outside $V(P)$. Since the colouring χ is strongly proper, the colours of these edges form a matching of $(r-2)$-sets of size more than $\ell(r-1)$. Let $C(P) = \bigcup_{e \in E(P)} \{c | c \in \chi(e)\}$. Then $|C(P)| \leq j(r-2) < \ell(r-2)$. Hence, there must exist a vertex $v_{j+1} \in V(F)$ outside $V(P)$ such that $\chi(v_j v_{j+1}) \cap C(P) = \emptyset$. Now, $P \cup v_j v_{j+1}$ is a longer good path than P, a contradiction. ∎

If $m = 0$ (i.e., $L = F$), then since H is connected and by the choice of L, there exists some edge in E_1 (which is also a good path) with an endpoint in $L = F$. So the conclusion follows by Claim 4.3. Hence we may assume that $m \geq 1$.

Let us call a path $x_{j_1} x_{j_2} \dots x_{j_t}$ in L an *increasing path* under σ if $x_{j_1} < x_{j_2} < \dots < x_{j_t}$ in σ; we call x_{j_t} the *last vertex* of the path. Let $\mathcal{P}$ be the collection of strongly rainbow increasing paths $x_{j_1} x_{j_2} \dots x_{j_t}$ in L satisfying that either (a) $2 \leq t \leq \ell$, the last vertex is in F and all other vertices are not, or (b) $t = \ell$ and all vertices are not in F. We note that $\mathcal{P} \neq \emptyset$, as L is connected and thus there is an edge connecting $\{x_1, ..., x_m\}$ and $V(F)$. Among all the paths in $\mathcal{P}$, let $P = x_{j_1} x_{j_2} \dots x_{j_t}$ be such that j_1 is minimum. By our assumption, $j_1 \in [m]$ and so $|N_L(x_{j_1}) \cap \{x_{j_1+1}, \dots, x_{|V(L)|}\}| < r\ell$. If $|N_L(x_{j_1}) \cap \{x_1, \dots, x_{j_1-1}\}| > \ell(r-2)$ then by a similar argument as in the proof of Claim 4.3 we can find $j_0 < j_1$ such that $\chi(x_{j_0} x_{j_1})$ is disjoint from all $\chi(x_{j_i} x_{j_{i+1}})$ for all $i \in [t-1]$ and $x_{j_0} x_{j_1} \in L$. In this case, either $x_{j_0} x_{j_1} \dots x_{j_{t-1}}$ or $x_{j_0} x_{j_1} \dots x_{j_t}$ would contradict our choice of P. Hence, $|N_L(x_{j_1}) \cap \{x_1, \dots, x_{j_1-1}\}| \leq \ell(r-2)$. This shows that $d_L(x_{j_1}) < \ell(r-2) + r\ell$. Since $\delta_H(x_{j_1}) \geq 4r\ell$, x_{j_1} is incident to more than $4\ell r - \ell r - \ell(r-2) = 2\ell(r+1)$ many edges in E_1. Among them, more than $2\ell r + \ell$ of them joins x_{j_1} to a vertex outside $V(P)$. Since χ is strongly proper, the colours on these edges form a matching of size more than $2\ell r + \ell$. Since $C := \bigcup_{i=1}^{t} \chi(x_{j_i} x_{j_{i+1}})$ has size less than ℓr, there must exist at least one edge of E_1 that joins x_{j_1} to a vertex x_{j_0} outside $V(P)$ such that $\chi(x_{j_0} x_{j_1})$ is disjoint from C. Now, $x_{j_0} x_{j_1} \dots x_{j_t}$ is a good path of length t. If $t = \ell$ then we are done. Otherwise $t < \ell$, then $x_{j_t} \in V(F)$ and this lemma follows by Claim 4.3. ∎

The following cleaning lemma is similar to part of Lemma 3.4.

Lemma 4.4 *Let H be a linear r-partite r-graph with an r-partition $(A_1, \cdots, A_r)$. Let M be an $(r-1)$-uniform matching where for each $f \in M$, f contains one vertex of each of $A_2, \dots, A_r$. Then there exists a subgraph $H' \subseteq H$ such that*

(1) $e(H') \geq [1/(r-1)]^{r-1} e(H)$,

(2) each edge of M intersects $V(H')$ in at most one vertex.

Proof. Let us independently colour each edge of M using a colour in $\{2, \dots, r\}$ chosen uniformly at random. Denote the colouring c. For each $i \in \{2, \dots, r\}$, let $M_i = \{f \in M : c(f) = i\}$ and let $B_i = \{f \cap A_i : f \in M_i\}$. Let H' be the subgraph of H induced by the edge set $\{e \in E(H) : e \cap V(M) \subseteq B_2 \cup \dots \cup B_r\}$. Note that $V(H') \subseteq (V(H) \backslash V(M)) \cup (B_2 \cup ... \cup B_r)$.

Let f be any edge of M. Suppose f is coloured i. Then since M is a matching, we have $|f \cap B_i| = 1$ and $f \cap B_j = \emptyset$ for each $j \in \{2, \dots, r\} \setminus \{i\}$. Therefore, $|f \cap V(H')| = 1$ and this proves (2).

Next, we claim that for any edge e in H, the probability that e is in H' is at least $[1/(r-1)]^{r-1}$. Let $s = |e \cap V(M)|$. If $s = 0$ then e is in H' with probability 1. So we may assume that $1 \leq s \leq r-1$. Since G is r-partite, the s vertices of S all lie in different parts among $A_2, \dots, A_r$. Without loss of generality, suppose $e \cap V(M) = \{a_2, \dots, a_{s+1}\}$, where $a_i \in A_i$ for each $i = 2, \dots, s+1$. Since M is matching, for each $i = 2, \dots, s+1$, there is a unique edge $f_i \in M$ that contains a_i. The probability that $a_i \in B_i$ is the probability that f_i is coloured i, which is $1/(r-1)$. Hence, the probability that $a_i \in B_i$ for each $i \in \{2, \ldots, s+1\}$ is $[1/(r-1)]^s$. In other words, the probability that e is in H' is $[1/(r-1)]^s \geq [1/(r-1)]^{r-1}$, as claimed. So the expectation of $e(H')$ is at least $[1/(r-1)]^{r-1}e(H)$. Therefore, there exists a colouring c for which $e(H') \geq [1/(r-1)]^{r-1}e(H)$. This subgraph H' satisfies the requirements of the lemma. ■

4.2 Rooted expanded trees and linear cycles of consecutive even lengths

In this subsection, we introduce some of the key notions we use, in particular, a variant of a breadth-first-search tree in a linear r-partite r-graph G, and prove some auxiliary results we need for the proof of the main theorem.

Definition 4.5 *Let $r \geq 3$ be an integer. Let G be a graph. Let χ be any edge-colouring of G by $(r-2)$-sets satisfying that for every edge $uv \in E(G)$ we have $u, v \notin \chi(uv)$. We define the (χ, r)-*expansion *of G, denoted by G^χ, to be the r-graph on vertex set $V(G) \cup \chi(G)$ obtained from G by expanding each edge e of G into the r-set $e \cup \chi(e)$, where $\chi(G) = \{c \in \chi(e)$ for some $e \in E(G)\}$.*

In the definition of (χ, r)-expansion we don't require the sets $V(G)$ and $\chi(G)$ to be disjoint. However, if χ is a strongly rainbow, then (χ, r)-expansion is isomorphic to what is known in the literature, as the r-expansion *of G, defined as follows. The r-expansion G^r of G is an r-graph obtained from G by expanding each edge e of G into an r-set using pairwise distinct $(r-2)$-sets disjoint from $V(G)$. Note that the $(r-2)$-sets used for the r-expansion naturally define a strongly rainbow edge-colouring on G.*

Algorithm 4.6 (Maximal Expanded Rooted Tree - MERT)

Input: A linear r-partite r-graph G with a fixed r-partition $(A_1, \ldots, A_r)$ and a vertex x in A_1.

Output: (H, T, χ) where H is some subgraph of G, T is a tree rooted at x such that H is the r-expansion of T and furthermore, for each $i \geq 0$, there exists some $j \in [r]$ such that $L_i(x) \subseteq A_j$, where $L_i(x)$ is the ith level in T, and finally χ is a strongly rainbow edge-colouring of T.

We will also obtain a collection of subgraphs of H, $\{H_i\}_{i=0}^m$ where each H_i is called the ith segment of H and a collection of $(r-1)$-uniform matchings $\{M_i\}_{i=1}^{m-1}$ where $V(M_i) \subset V(H_{i+1}) \setminus V(H_i)$ and M_i is called the ith matching of H, these are described further below.

Initialization: Let $H_0 = \{x\}$. Let $L_0 = \{x\}$ and $T_0 = \{x\}$. Let H_1 be the subgraph of G consisting of all edges of G containing x. For every $v \in E(H_1) \setminus \{x\}$, let $p_v = \{x\}$. Finally, let $i = 1$.

Iteration: Let E_i denote the set of edges in G that contain exactly one vertex in $V(H_i) \setminus V(H_{i-1})$ and no vertices in $\cup_{j<i} V(H_j)$.

If $E_i = \emptyset$ then let $L_i = (V(H_i) \setminus V(H_{i-1})) \cap A_j$, where A_j is any part of $(A_1, \ldots, A_r)$ that doesn't contain L_{i-1}. Let T_i be the super-tree of T_{i-1} obtained from T_{i-1} by joining every $v \in L_i$ to $p_v \in L_{i-1}$. Let $H = \cup_{0 \leq j \leq i} H_i$, $T = T_i$ and terminate.

If $E_i \neq \emptyset$ then do the following. Suppose $L_{i-1} \subseteq A_\ell$. For each $j \in [r] \setminus \{\ell\}$, let E_i^j be the set of edges in $e \in E_i$ such that $|e \cap (V(H_i) \setminus V(H_{i-1})) \cap A_j| = 1$. Then $E_i = \bigcup_{j \in [r] \setminus \{\ell\}} E_i^j$. Let $s(i)$ be some $j \in [r] \setminus \{\ell\}$ that maximizes $|E_i^j|$. Let $L_i = E_i^{s(i)} \cap A_{s(i)}$. Let T_i be the super-tree of T_{i-1} obtained from T_{i-1} by joining every $v \in L_i$ to $p_v \in L_{i-1}$. Let M_i be a largest matching of $(r-1)$-tuples in $\{e \setminus L_i : e \in E_i^{s(i)}\}$. For each $I \in M_i$ we do the following. Since the graph G is linear, there is a unique $v_I \in L_i$ such that $I \cup v_I \in E_i^{s(i)}$. For each $u \in I$, we define p_u to be v_I and refers to it as the *parent* of u. Let H_{i+1} be the subgraph of G induced by the edge set $\{I \cup v_I | I \in M_i\}$. Increase i by one and repeat.

Termination: Suppose the algorithm stopped after m steps then we call m the *height* of H, noting that m is also the height of the tree T. We will interchangeably call both H and the pair (H, T) an *MERT* of G rooted at x. Let χ be the following colouring on $E(T)$: For every edge $uv \in E(T)$ there is a unique $(r-2)$-tuple I such that $uv \cup I \in E(H)$, we let $\chi(uv) = I$. By construction of H, χ is strongly rainbow. ∎

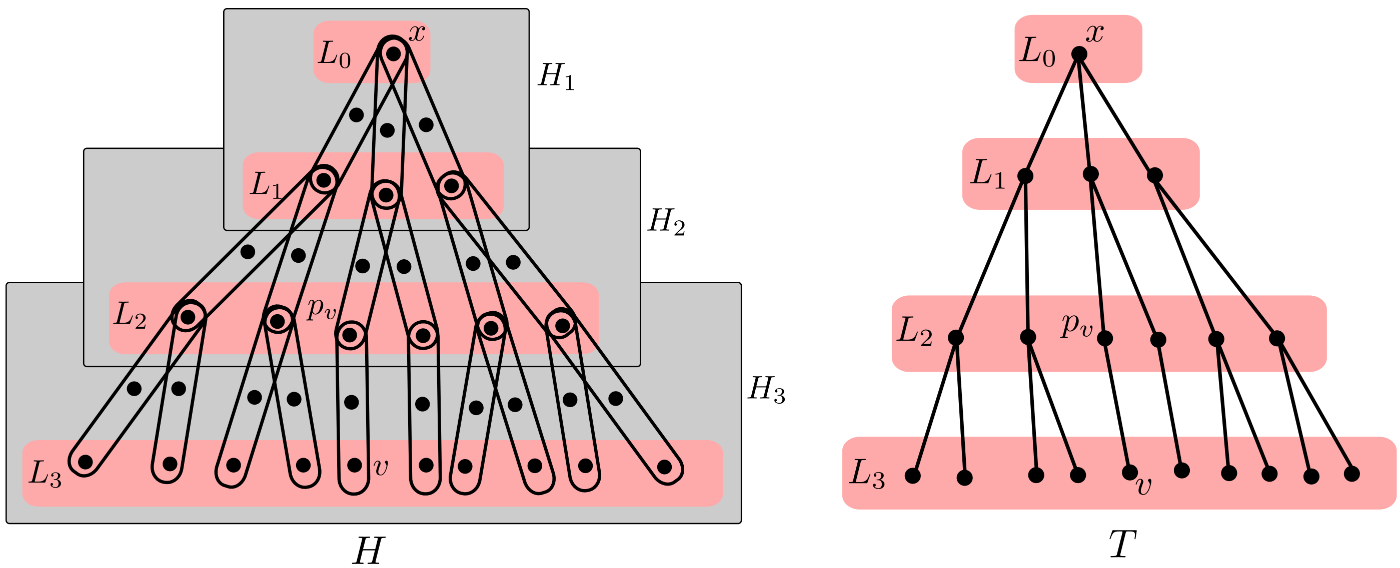

Figure 1: $H = H_1 \cup H_2 \cup H_3$ and the corresponding tree T

Lemma 4.7 *Let $r \geq 3$ and $k, t \geq 1$ be integers. Let G be an r-partite r-graph with an r-partition $(A_1, A_2, \ldots, A_r)$. Let x be a vertex in G. Let (H, T) be an MERT rooted at x. Let D be the subgraph of G consisting of all the edges in G that contain a vertex in L_{t-1}, at least one vertex in $V(H_t) \setminus L_{t-1}$ and no vertices from $\left(\bigcup_{j<t} V(H_j)\right) \setminus L_{t-1}$. If $e(D) \geq 8kr(r-1)(|L_{t-1}| + |L_t|)$ then G contains linear cycles of lengths $2\ell+2, 2\ell+4, \ldots, 2\ell+2k$ for some $\ell \leq t-1$.*

Proof. By definition of MERT, without loss of generality we may suppose $L_{t-1} \subseteq A_1$. By definition, each edge of D contains a vertex in L_{t-1} and at least one vertex in $V(H_t) \setminus L_{t-1}$. Since $A_2 \cap V(H_t), A_3 \cap V(H_t), \ldots, A_r \cap V(H_t)$ partition $V(H_t) \setminus L_{t-1}$, by the pigeonhole principle, for some $i \in \{2, \ldots, r\}$, at least $e(D)/(r-1)$ of the edges of D contain a vertex from $A_i \cap V(H_t)$. Without loss of generality, suppose $i = 2$.

Let $X = L_{t-1}$ and $Y = A_2 \cap V(H_t)$. By definition of MERT, we have $|Y| = |V(H_t) \cap A_2| = |L_t|$. Let D' be the subgraph of D consisting of the edges that contain a vertex in X and a vertex in Y. By the previous discussion,

$$e(D') \geq e(D)/(r-1). \tag{2}$$

Let B be the (X, Y)-projection of D'. Since G is linear, $e(B) = e(D')$. Also, $|V(B)| \leq |X| + |Y| = |L_{t-1}| + |L_t|$. By our assumption about $e(D)$ and (2),

$$e(B) = e(D') \geq 8kr(|L_{t-1} + |L_t|) \geq 8kr|V(B)|.$$

So B has average degree at least $16kr$. By Lemma 3.1, B contains a connected subgraph B' with minimum degree at least $8kr$.

Let $S = V(B') \cap X$. Suppose x' is the closest common ancestor of S in the tree T. The union of the paths of T joining vertices of S to x' forms a subtree T' of T rooted at x'. Suppose that $x' \in L_j$ for some $j \geq 0$. Then $V(T') \subseteq L_j \cup \dots \cup L_{t-1}$, and x' is the only vertex in $V(T') \cap L_j$. For each $v \in S$, let Q_v denote the unique (v, x')-path in T'.

Since x' is the closest common ancestor of S in T, x' has at least two children in T'. Let x_1 be one of the children of x in T'. We define a vertex labelling f on S as follows. For each $v \in S$, if Q_v contains x_1 then let $f(v) = 1$, and otherwise let $f(v) = 2$. Note that since x had at least two children, there will be some $u, v \in S$ with $f(u) = 1$ and $f(v) = 2$. The following claim is one of the key ingredients used by Bondy and Simonovits in proving their results in [3]. For completeness, we include a proof.

Claim 4.8 *Let $u, v \in S$. If $f(u) = 1$ and $f(v) = 2$ then $Q_u \cup Q_v$ is a path of length $2(t-1-j)$ in T' that intersects S only in u and v.*

Proof. By the definition of f, u lies in the subtree T_1 of T' under x_1, while v lies in a subtree T_2 of T' under a different child, say x_2 of x'; these two subtrees of T' are vertex disjoint. The path Q_u consists of the unique (x_1, u)-path in T_1 and $x'x_1$ and the path Q_v consists of the unique (x_2, v)-path in T_2 and $x'x_2$. Hence the two paths intersect only at x' and hence has length $2(t-1-j)$ in T'. It is also clearly that their union both intersects S in u and v. ∎

Now, we define a partition of $E(B')$ into E_1 and E_2 as follows. Let ab be any edge in $E(B')$ where $a \in X$ and $b \in Y$. For $i = 1, 2$, we put ab in E_i if $f(a) = i$. We define an edge-coloring φ on B' using $(r-2)$-sets by letting $\varphi(ab)$ be the unique $(r-2)$-set such that $ab \cup \varphi(ab) \in E(D')$ for all $ab \in E(B')$. Since G is r-partite, $\varphi(B^*)$ is disjoint from $V(B^*)$. Since G is linear, φ is strongly proper. We may assume that $|E_1| \leq |E_2|$. By Lemma 4.2, with $\ell = 2k$, B' contains a strongly rainbow path $P = a_1 b_1 a_2 b_2 \dots a_k b_k a_{k+1}$ of length $2k$ such that the first edge of P is in E_1 and all other edges are in E_2. Note that we must have $a_1 \in S$. Otherwise if $b_1 \in S$ instead then the first two edges of P would have the same colour, contradicting our definition of P. Hence, $a_1, a_2, \dots, a_{k+1} \in S$ and by our assumption about P, $f(a_1) = 1$ and $f(a_2) = \dots = f(a_{k+1}) = 2$. For each $i \in [k]$, let P_i be the subpath P from a_1 to a_i. Let χ be the colouring in (H, T, χ) produced by Algorithm 4.6.

Claim 4.9 *For each $i \geq 2$, let R_i be the union of the r-uniform paths P_i^{φ}, $Q_{a_1}^{\chi}$ and $Q_{a_i}^{\chi}$. Then R_i is a linear cycle of length $2(t-1-j) + 2(i-1)$ in G.*

Proof. Since $f(a_1) = 1$ and $f(a_i) = 2$, by Claim 4.8, $Q_{a_1} \cup Q_{a_i}$ is a path of length $2(t-1-j)$ in T' that intersects S only in a_1 and a_i. On the other hand, P_i is a path of length $2(i-1)$ in B', which intersects $Q_{a_1} \cup Q_{a_i}$ only at a_1 and a_i. So $P_i \cup Q_{a_1} \cup Q_{a_i}$ is a cycle of length $2(t-1-j) + 2(i-1)$ in $T' \cup B'$. By our assumptions, φ is strongly rainbow on P_i and χ is strongly rainbow on $Q_{a_1} \cup Q_{a_i}$. Furthermore, for any $e \in P_i$ and $f \in Q_{a_1} \cup Q_{a_i}$, $\varphi(e)$ has no vertices in $\left(\bigcup_{j<t} V(H_j)\right) \setminus L_{t-1}$, while $\chi(f) \subseteq \left(\bigcup_{j<t} V(H_j)\right) \setminus L_{t-1}$. So $\varphi(e) \cap \chi(f) = \emptyset$. Therefore, R_i is a linear cycle of length $2(t-1-j) + 2(i-1)$ in G. ∎

By Claim 4.9, we see Lemma 4.7 holds for some $\ell = t-1-j \leq t-1$. ∎

In the next lemma, we in fact obtain linear cycles of consecutive lengths, instead of just consecutive even lengths.

Lemma 4.10 *Let $r \geq 3$, and $k, t \geq 1$ be integers. Let G be an r-partite r-graph with an r-partition $(A_1, A_2, \ldots, A_r)$. Let x be a vertex in G. Let (H, T) be an MERT rooted at x. For any integer $t \geq 1$, let*

$$F = \{e \in E(G) : e \cap \bigcup_{i<t} V(H_i) = \emptyset \text{ and } |e \cap V(H_t)| \geq 2|\}.$$

If $e(F) \geq 8kr^{r+2}|L_t|$, then G contains linear cycles of lengths $2\ell + 1, 2\ell + 2, \ldots, 2\ell + 2k$ for some $\ell \leq t$.

Proof. By our assumption L_{t-1} is contained in one partite set of G. Without loss of generality suppose that $L_{t-1} \subseteq A_1$. By definition of the MERT, $M^* = \{e \setminus L_{t-1} : e \in H_t\}$ is an $(r-1)$-uniform matching contained in $A_2 \cup \cdots \cup A_r$. By Lemma 4.4, there exists a subgraph F' of F such that

1. $e(F') \geq (1/(r-1))^{r-1} e(F)$,
2. each edge of M^* intersects $V(F')$ in at most one vertex.

Since $V(F')$ is disjoint from L_{t-1}, item 2 above ensures that

$$\forall e \in H_t,\ |e \cap V(F')| \leq 1. \tag{3}$$

By the definition of F and the fact that $F' \subseteq F$, any edge in F' contains at least two vertices of $V(H_t) \backslash L_{t-1} = V(M^*)$. By the pigeonhole principle, there exist some $i, j \in \{2, \ldots, r\}$ such that the subgraph F'' of F' with edge set $E(F'') := \{e \in F' : |e \cap V(M^*) \cap A_i| = |e \cap V(M^*) \cap A_j| = 1\}$ satisfies

$$e(F'') \geq e(F') / \binom{r-1}{2} \geq (2/r^{r+1}) e(F) \geq 16kr|L_t|, \tag{4}$$

where the last inequality holds as $e(F) \geq 8kr^{r+2}|L_t|$.

Without loss of generality, suppose that $\{i, j\} = \{2, 3\}$. Let B be the (A_2, A_3)-projection of F''. Since G is linear, $e(B) = e(F'')$. Also, note that $|V(B)| \leq |V(M^*) \cap (A_2 \cup A_3)| \leq 2|L_t|$. Hence, by (4),

$$e(B) = e(F'') \geq 16kr|L_t| \geq 8kr|V(B)|.$$

So B has average degree at least $16kr$. By Lemma 3.1, B contains a connected subgraph B^* such that

$$\delta(B^*) \geq 8kr.$$

For each vertex $y \in V(B^*)$, there is a unique edge e_y of H_t that contains y. Let v_y be the unique vertex in $e_y \cap L_{t-1}$. Let $S = \{v_y : y \in V(B^*)\}$. Let x' be the closest common ancestors of S in T. Let T' be the subtree formed by the paths in T from S to x'. Suppose that $x' \in L_j$. Then $V(T') \subseteq L_j \cup \cdots \cup L_{t-1}$ and that x' is the only vertex in $V(T') \cap L_j$. Since x' is the closest common ancestor of S in T, x' has at least two children in T'. For each $v \in S$, let Q_v denote the unique (v, x')-path in T'.

Now we define a labelling f of vertices in S as follows. Let x_1 be one child of x in T'. For each $v \in S$, if Q_v contains x_1 then let $f(v) = 1$; otherwise let $f(v) = 2$. As in the proof of Lemma 4.7, the definitions of T' and f ensure the following.

Claim 4.11 *Let $u, v \in S$. If $f(u) = 1$ and $f(v) = 2$, then $Q_u \cup Q_v$ is a path of length $2(t - 1 - j)$ in T' that intersects S only in u and v.* ■

We now partition $E(B^*)$ into M and N as follows. Let

$$M = \{ab \in E(B^*) : f(v_a) = f(v_b)\} \quad \text{and} \quad N = \{ab \in E(B^*) : f(v_a) \neq f(v_b)\}.$$

Let us define an edge-colouring φ of B^* using $(r-2)$-sets as follows. For all $ab \in E(B^*)$, let $\varphi(ab)$ be the unique $(r-2)$-set such that $ab \cup \varphi(ab) \in E(F'') \subseteq E(G)$. Since G is r-partite, $\varphi(B^*)$ is disjoint from $V(B^*)$. Since G is linear, φ is strongly proper. There are two cases to consider.

Case 1. $|M| \geq |N|$.

Applying Lemma 4.2 with $E_1 = N, E_2 = M$, $\ell = 2k$, there exists a strongly rainbow path (under φ) $P = ab_1b_2 \dots b_{2k}$ of of length $2k$ in B^* such that the first edge is in N and all the other edges are in M. Let us assume that $f(v_a) = 1$; the case $f(v_a) = 2$ can be argued similarly. Since $ab_1 \in N$ and $b_ib_{i+1} \in M$ for all $i \in [2k-1]$, we have $f(v_{b_1}) = \cdots = f(v_{b_{2k}}) = 2$. Consider any $i \in [2k]$. Let P_i denote the portion of P between a and b_i. Since φ is strongly rainbow on P_i and $\varphi(P_i) \subseteq V(F'')$, P_i^φ is a linear path of length i in F''. Note that $f(v_a) \neq f(v_{b_i})$. By Claim 4.11, $Q_{v_a} \cup Q_{v_{b_i}}$ is a path of length $2(t-1-j)$ in T'. Since χ is strongly rainbow on T', $Q^\chi_{v_a} \cup Q^\chi_{v_{b_i}}$ is a linear path of length $2(t-1-j)$ in $\bigcup_{j<t} H_j$. In particular, we see that P_i^φ and $Q^\chi_{v_a} \cup Q^\chi_{v_{b_i}}$ are vertex disjoint. By (3) and the fact that $e_a, e_{b_i} \in H_t$ are disjoint, one can easily check that $R_i := P_i^\varphi \cup Q^\chi_{v_a} \cup Q^\chi_{v_{b_i}} \cup \{e_a, e_{b_i}\}$ is a linear cycle of length $2(t-j)+i$ in G. This gives linear cycles of lengths $2\ell+1, 2\ell+2, \dots, 2\ell+2k$ in G, where $\ell = t-j \leq t$, as desired.

Case 2. $|N| \geq |M|$.

In this case, we apply Lemma 4.2 with $E_1 = M$, $E_2 = N$, $\ell = 2k$. There exists a strongly rainbow path $a'ab_1b_2 \dots b_{2k-1}$ of length $2k$ in B' such that the first edge is in M and all the other edges are in N. Without loss of generality, suppose $f(v_{a'}) = f(v_a) = 1$. Since $ab_1 \in N$ and $b_ib_{i+1} \in N$ for each $i \in [2k-2]$, we have $f(v_{b_1}) = f(v_{b_3}) = \cdots = f(v_{b_{2k-1}}) = 2$. By the same reasoning as in Case 1, for each $i \in [k]$, we can use the strongly rainbow path $ab_1 \dots b_{2i-1}$ of length $2i-1$ to find a linear cycle of length $2(t-j)+(2i-1)$ in G. Also we can use the strongly rainbow path $a'ab_1 \dots b_{2i-1}$ to build a linear cycle of length $2(t-j)+2i$ in G for each $i \in [k]$. Together, these give us linear cycles of length $2\ell+1, 2\ell+2, \dots, 2\ell+2k$, where $\ell = t-j \leq t$. This proves Lemma 4.10. ■

4.3 Linear cycles of even consecutive lengths in linear r-graphs

Now we develop our main result for the section. Our result is that for each $r \geq 3$ there are constants c_3, c_4, depending only on r such that in every n-vertex linear r-graph G with average degree $d(G) \geq c_3k$ we can find linear cycles of lengths $2\ell+2, 2\ell+4, \dots, 2\ell+2k$ for some $\ell \leq \lceil \frac{\log n}{\log(d(G)/k)-c_2} \rceil - 1$. This would also immediately yield an improved bound on the linear Turán number of an r-uniform linear $2k$-cycle.

Definition 4.12 Given a positive real d, an r-graph G is said to be *d-minimal*, if $d(G) \geq d$ but for every proper induced subgraph H we have $d(H) < d(G)$.

Lemma 4.13 *Let $r \geq 2$ be an integer and $d > 0$ a real. If G is an r-graph satisfying that $d(G) \geq d$ then G contains a d-minimal subgraph G'.*

Proof. Among all induced subgraphs H of G satisfying $d(H) \geq d$, let G' be one that minimizes $|V(G')|$. Then G' is d-minimal. ■

Lemma 4.14 *Let $r \geq 3$ be an integer and d a positive real. Let G be a d-minimal r-graph. For any proper subset S of $V(G)$, the number of edges of G that contain a vertex in S is at least $d|S|/r$.*

Proof. Otherwise, suppose there is a proper subset S of $V(G)$ such that the number of edges of G that contain a vertex in S is at most $d|S|/r$. Then the subgraph G' of G induced by $V(G) \setminus S$ satisfies

$$e(G') \geq e(G) - d|S|/r \geq d|V(G)|/r - d|S|/r = d(|V(G')|/r.$$

Hence $d(G') \geq d$, contradicting G being d-minimal. ∎

Theorem 4.15 *Let k, r be integers where $k \geq 1$ and $r \geq 3$. Let $c_3 = 128r^{r+3}$ and $c_4 = \log(64r^{r+2})$. If G is an n-vertex r-partite linear r-graph with average degree $d(G) \geq c_3 k$, then G contains linear cycles of lengths $2\ell+2, 2\ell+4, \ldots, 2\ell+2k$, for some positive integer $\ell \leq \lceil \frac{\log n}{\log(d(G)/k) - c_4} \rceil - 1$.*

Proof. Let $d = d(G)$ and let $p = \lceil \frac{\log n}{\log(d/k) - c_4} \rceil$. By Lemma 4.13, G contains a d-minimal subgraph G'. Suppose G' does not contain a collection of linear cycles of lengths $2\ell+2, 2\ell+4, \ldots, 2\ell+2k$, where $\ell \leq p-1$. We will derive a contradiction. Let us apply Algorithm 4.6 to G' with a fixed vertex x and let (H, T, χ) be the triple produced. Let m denote the height of H and T.

For each $i \in [m]$, let

$$G_i = \{e \in E(G') \setminus E(H) : e \cap V(H_i) \neq \emptyset, e \cap \bigcup_{j<i} V(H_j) = \emptyset\},$$

$$G_i^1 = \{e \in E(G_i) : |e \cap V(H_i)| = 1\}, \quad \text{and} \quad F_i = \{e \in E(G_i) : |e \cap V(H_i)| \geq 2\}.$$

Note that $G_m^1 = \emptyset$, as otherwise Algorithm 4.6 would have produced non-empty L_{m+1}, instead of stopping at step m, L_m being the last level. For convenience, define $L_{m+1} = \emptyset$.

Claim 4.16 *For each $1 \leq i \leq \min\{m, p\} - 1$, we have $e(G_i^1) \leq 8kr^3(|L_i| + |L_{i+1}|)$.*

Proof. Let D_i be the set of edges in G_i^1 that intersect $V(H_i)$ in L_i. By Algorithm 4.6,

$$e(D_i) \geq e(G_i^1)/r.$$

Let $e \in D_i$. By definition, e intersects $V(H_i)$ in exactly one vertex and that vertex lies in L_i. Furthermore, e contains no vertex in $\bigcup_{j<i} V(H_j)$. If $e \setminus L_i$ is vertex disjoint from $V(H_{i+1}) \setminus L_i$, then e would have been added to H_{i+1} by Algorithm 4.6, contradicting $e \notin E(H)$. Hence e must contain at least one vertex in $V(H_{i+1} \setminus L_i)$. If $e(D_i) \geq 8kr(r-1)(|L_i|+|L_{i+1}|)$ then by Lemma 4.7 (with $t = i+1$) G contains linear cycles of lengths $2\ell+2, 2\ell+4, \ldots, 2\ell+2k$ for some $\ell \leq i \leq \min\{m, p\} - 1 \leq p-1$, contradicting our assumption. Hence,

$$e(D_i) \leq 8kr(r-1)(|L_i| + |L_{i+1}|) < 8kr^2(|L_i| + |L_{i+1}|).$$

Therefore, we have $e(G_i^1) \leq 8kr^3(|L_i| + |L_{i+1}|)$. ∎

Claim 4.17 *For each $1 \leq i \leq \min\{m, p-1\}$ we have $e(F_i) \leq 8kr^{r+2}|L_i|$.*

Proof. Suppose $e(F_i) \geq 8kr^{r+2}|L_i|$. Then by Lemma 4.10 (with $t = i$), we can find in G linear cycles of length $2\ell+2, 2\ell+4, \ldots, 2\ell+2k$ for some $\ell \leq i \leq p-1$, contradicting our assumption. Hence, $e(F_i) \leq 8kr^{r+2}|L_i|$ hods for each $1 \leq i \leq \min\{m, p-1\}$. ■

By Claims 4.16 and 4.17, we have that for any $1 \leq i \leq \min\{m,p\}-1$,

$$e(G_i) = e(G_i^1) + e(F_i) \leq 16kr^{r+2}(|L_i| + |L_{i+1}|) \text{ and thus } e(\bigcup_{j=1}^{i} G_j) \leq \sum_{j=1}^{i} 16kr^{r+2}(|L_j| + |L_{j+1}|). \quad (5)$$

Claim 4.18 *For each* $1 \leq i \leq \min\{m,p\}-1$, $e(\bigcup_{j=1}^{i} G_j) \geq (d/2)\sum_{j=1}^{i} |L_j| - |L_{i+1}|$.

Proof. Let $S = \bigcup_{j=0}^{i} V(H_j)$. Since $i \leq m-1$, S is a proper subset of $V(G')$. Let E_S denote the set of edges of G' that contains a vertex in S. By our definitions, $E_S \subseteq \bigcup_{j=1}^{i+1} E(H_j) \cup \bigcup_{j=1}^{i} G_j$. Since G' is d-minimal, by Lemma 4.14,

$$|E_S| \geq d|S|/r = d\big(1 + \sum_{j=1}^{i}(r-1)|L_j|\big)/r.$$

On the other hand, by the definition of H, $|\bigcup_{j=1}^{i+1} E(H_j)| = \sum_{j=1}^{i+1} |L_j|$. Hence, we have

$$\begin{aligned} e(\bigcup_{j=1}^{i} G_j) &= |E_S| - |\bigcup_{j=1}^{i+1} E(H_j)| \geq d\big(1 + \sum_{j=1}^{i}(r-1)|L_j|\big)/r - \sum_{j=1}^{i+1} |L_j| \\ &\geq \sum_{j=1}^{i} |L_j|\big(d(1-1/r)-1\big) + d/r - |L_{i+1}| \geq (d/2)\sum_{j=1}^{i} |L_j| - |L_{i+1}|, \end{aligned}$$

completing the proof. ■

For each $1 \leq i \leq m$, let $U_i = \bigcup_{j=1}^{i} L_j$. By (5) and Claim 4.18, for each $0 \leq i \leq \min\{m,p\}-1$,

$$32kr^{r+2}|U_{i+1}| - |L_{i+1}| \geq \sum_{j=1}^{i} 16kr^{r+2}(|L_j| + |L_{j+1}|) \geq e(\bigcup_{j=1}^{i} G_j) \geq (d/2)|U_i| - |L_{i+1}|.$$

Hence, for each $0 \leq i \leq \min\{m,p\}-1$ we have

$$|U_{i+1}| \geq (d/64kr^{r+2})|U_i|. \quad (6)$$

Claim 4.19 $m \geq p$.

Proof. Suppose otherwise that $m \leq p-1$. Let $S = V(H_m) \setminus (L_{m-1} \cup L_m)$. Then S is a proper subset of $V(G')$ with $|S| = (r-2)|L_m|$. Let E_S denote the set of edges of G' that contain a vertex in S. Since G' is d-minimal, we have

$$|E_S| \geq d|S|/r = d|L_m|(r-2)/r.$$

On the other hand, since L_m is the last level of H, by the definitions and the fact that $G_m^1 = \emptyset$, we have $E_S \subseteq E(H_m) \cup \bigcup_{i=1}^{m-1} E(G_i) \cup F_m$. By (5), Claim 4.17 and the fact that $e(H_m) = |L_m|$, we have

$$|E_S| \leq |L_m| + \sum_{j=1}^{m-1} 16kr^{r+2}(|L_j| + |L_{j+1}|) + 8kr^{r+2}|L_m| \leq 32kr^{r+2}|U_{m-1}| + 16kr^{r+2}|L_m|.$$

Combining the lower and upper bounds above on $|E_S|$, we get

$$(r-2)d|L_m|/r \leq 32kr^{r+2}|U_{m-1}| + 16kr^{r+2}|L_m|.$$

As $d \geq c_3 k = 128kr^{r+3}$, we have $d/r \geq 128kr^{r+2}$. This inequality above implies $|L_m| < |U_{m-1}|$ and thus $|U_m| = |U_{m-1}| + |L_m| < 2|U_{m-1}|$. But by (6), we have

$$|U_m| \geq (d/64kr^{r+2})|U_{m-1} \geq 2|U_{m-1}|,$$

a contradiction. ■

Using $m \geq p$, now we show that the expansion rate was so fast that $|U_p| \geq n$, which would lead to a contradiction. Recall that $|U_0| = |L_0| = 1$. Thus by (6), we have

$$|U_p| \geq (d/64kr^{r+2})^p.$$

Taking logarithm of both sides of the inequality and using $c_4 = \log(64r^{r+2})$ and $p = \lceil \frac{\log n}{\log(d/k)-c_4} \rceil$, we get

$$\log|U_p| \geq p(\log(d/k) - c_4) \geq \log n.$$

So $|U_p| \geq n$. This is a contradiction as $x \notin U_p$, completing the proof of the theorem. ■

Finally we are ready to prove Theorem 1.3. We need the following result of Erdős and Kleitman.

Lemma 4.20 [10] *Let $r \geq 2$. Every r-graph G contains an r-partite subgraph G' with $e(G') \geq (r!/r^r)e(G)$.*

Proof of Theorem 1.3: Let $r \geq 3$ and $k \geq 2$ be the given integers. Let c_3, c_4 be the constants obtained in Theorem 4.15. Let $c_1 = c_3 r^r = 128r^{2r+3}$ and $c_2 = c_4 + \log(r^r) = \log(64r^{2r+2})$. Let G be an n-vertex r-graph with $d(G) \geq c_1 k$. By Lemma 4.20, G contains an r-partite subgraph G' with $d(G') \geq d(G)(r!/r^r) \geq d(G)/r^r \geq c_3 k$. By Theorem 4.15, G' (and thus G also) contains linear cycles of lengths $2\ell+2, 2\ell+4, \ldots, 2\ell+2k$, for some positive integer

$$\ell \leq \left\lceil \frac{\log n}{\log(d(G')/k) - c_4} \right\rceil - 1 \leq \left\lceil \frac{\log n}{\log(d(G)/k) - \log r^r - c_4} \right\rceil - 1 = \left\lceil \frac{\log n}{\log(d(G)/k) - c_2} \right\rceil - 1.$$

This finishes the proof of Theorem 1.3. ■

As mentioned in the introduction, as a quick application of Theorem 1.3, we obtain an improvement (in Corollary 1.4) on the bound given in [6] on the linear Turán number of an even cycle by reducing the coefficient from at least exponential in k to a function linear in k (for fixed r).

Proof of Corollary 1.4: Let $r \geq 3$ and $k \geq 2$ be the given integers. Let $c_1 = 128r^{2r+3}$ and $c_2 = \log(64r^{2r+2})$, as in the proof of Theorem 1.3. Let G be an n-vertex r-graph with $e(G) \geq 64kr^{2r+3}n^{1+1/k}$. Then $d(G) \geq 64kr^{2r+4}n^{1/k} \geq c_1 k$, thus we can apply Theorem 1.3 to G and conclude that it contains linear cycles of lengths $2\ell, 2\ell+4, \ldots, 2\ell+2k-2$ for some positive integer

$$\ell \leq \left\lceil \frac{\log n}{\log(d(G)/k) - c_2} \right\rceil \leq \left\lceil \frac{\log n}{\log(64r^{2r+4}) + \log n^{1/k} - c_2} \right\rceil \leq k.$$

Therefore the even numbers in the interval $[2\ell, \ldots, 2\ell+2(k-2)]$ contain the number $2k$, which means G' must contain a linear cycle of length exactly $2k$. Hence the corollary holds with $c = 64kr^{2r+3}$. ■

5 Concluding remarks

For fixed $r \geq 3$, by applying $k = 2$ to our main theorems, we conclude that there is a constant $c = c(r)$ such that in n-vertex linear r-graphs G with $d(G) \geq c$, there exist both an even linear cycle of length $O(\log n/(\log d(G))$ and an odd linear cycle of length $O(\log n/\log d(G))$, which is interesting on its own.

In Theorem 1.5, we can slightly improve the leading coefficient 6 on the bound of the shortest length of a cycle in the collection guaranteed to $3 + 2\sqrt{2}$ by choosing $d = \Theta_r(\delta^{\sqrt{2}-1}k^{2-\sqrt{2}})$ in our proof. However, we do not know if one can further improve the bound on the shortest length to $2\lceil \frac{\log n}{\log(d(G)/k - c_2} \rceil + c_3$, for some constants c_2, c_3, as in Theorem 1.3. In particular, we pose the following two questions, the second being a weakening of the first.

Question 5.1 *Let $r \geq 3$ and $k \geq 2$ be integers. Is it true that there exist constants $c_1 = c(r), c_2 = c(r)$ such that if G is an n-vertex linear r-graph with average degree $d(G) \geq c_1 k$ then G contains linear cycles of k consecutive lengths, the shortest of which is at most $2\lceil \frac{\log n}{\log d(G)/k - c_2} \rceil + 1$?*

Question 5.2 *Let $r \geq 3$ and $k \geq 2$ be integers. Is it true that there exist constants $c_1 = c(r), c_2 = c(r)$ such that if G is an n-vertex linear r-graph with average degree $d(G) \geq c_1 k$ then G contains linear cycles of k consecutive odd lengths, the shortest of which is at most $2\lceil \frac{\log n}{\log(d(G)/k) - c_2} \rceil + 1$?*

Ergemlidze, Győri and Methuku [14] proved that for $m \in \{2, 3, 4, 6\}$, $\mathrm{ex}_L(n, \{C_3^3, C_5^3, \cdots, C_{2m+1}^3\}) = \Omega(n^{1+1/m})$. Hence, for $m \in \{2, 3, 4, 6\}$ and all sufficiently large n, there are n-vertex linear 3-graphs G with $d(G) \geq cn^{1/m}$ that contain no odd linear cycles of length at most $2m + 1$. For these graphs, the bound on the shortest odd cycle length in Questions 5.1 and 5.2 is $2m + 3$ and hence would be best possible.

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6 Appendix

Proof of Proposition 1.6: A partial (n,k,q)-*Steiner system* is a family $\mathcal{F}$ of k-subsets on $[n]$ such that every q-subset of $[n]$ is in at most one member of $\mathcal{F}$. In particular, a partial $(n,k,2)$ Steiner system is a linear hypergraph. Rödl [30] showed that for all fixed $k > q \geq 2$, as $n \to \infty$ there exist partial (n,k,q)-Steiner systems of size $(1-o(1))\binom{n}{q}/\binom{k}{q}$ (see [27], [19] for recent breakthroughs on the existence of steiner systems). Let $m = \lfloor (1-\epsilon)\log_d n \rfloor$. By our discussion above, we can find a large enough integer n_0 such that for all $n \geq n_0$ there exists an n-vertex partial $(n,r,2)$-Steiner system G of size at least $0.9\binom{n}{2}/\binom{r}{2}$ and that the following inequality also holds

$$0.8dn^{\epsilon^2} > 2^{m+1}r. \tag{7}$$

By definition, G is a linear r-graph. Set $p = 2rd/n$ and let F be a random subgraph of G obtained by independently including each edge of G with probability p. Let $\mathbb{X}$ denote the number of edges in F and $\mathbb{Y}$ the number of linear cycles of length at most m in F. Then

$$\mathbb{E}[\mathbb{X}] \geq 0.9\binom{n}{2}/\binom{r}{2} \cdot (2rd/n) > 1.8dn/r.$$

On the other hand, observe that for any fixed ℓ, there are fewer than n^ℓ ways to choose a cyclic list $v_1v_2\dots v_\ell v_1$. Since G is linear, for each cyclic list $v_1v_2\dots v_\ell v_1$ there is at most one linear cycle in G with $v_1v_2\dots v_\ell v_1$ being its skeleton. So there are fewer than n^ℓ linear cycles of length ℓ in G. Hence, using $d \geq (2r)^{\frac{1}{\epsilon^2}}$ and $m \leq (1-\epsilon)\log_d n$, we have

$$\mathbb{E}[\mathbb{Y}] \leq \sum_{\ell=3}^{m} n^\ell (2rd/n)^\ell = \sum_{\ell=3}^{m} (2rd)^\ell < 2(2rd)^m < 2^{m+1}d^{(1+\epsilon)m} \leq 2^{m+1}n^{1-\epsilon^2}$$

Therefore, by (7),

$$\mathbb{E}[\mathbb{X}-\mathbb{Y}] > \frac{1.8dn}{r} - 2^{m+1}n^{1-\epsilon^2} > \left(\frac{1.8d}{r} - \frac{2^{m+1}}{n^{\epsilon^2}}\right)n \geq \frac{dn}{r}.$$

Hence there exists an F for which $\mathbb{X} - \mathbb{Y} \geq \frac{dn}{r}$. From F let us delete one edge from each linear cycle of length at most m. Let H be the remaining graph. Then H is an n-vertex linear r-graph that has average degree at least d and has no linear cycles of length at most $(1-\epsilon)\log_d n$. ■