

Discrepancies of perfect matchings in hypergraphs

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Abstract

In this paper, we determine the minimum degree threshold of perfect matchings with high discrepancy in r -edge-colored k -uniform hypergraphs for all $k \geq 3$ and $r \geq 2$, thereby completing the investigation into discrepancies of perfect matchings that has recently attracted significant attention. Our approach identifies this discrepancy threshold with a novel family of multicolored uniform hypergraphs and reveals new phenomena not covered in previous studies. In particular, our results address a question of Balogh, Treglown and Zárate-Guerén concerning 3-uniform hypergraphs.

1 Introduction

A *hypergraph* H consists of a vertex set $V(H)$ and an edge set $E(H)$ whose members are subsets of $V(H)$. For a positive integer k and a set S , let $\binom{S}{k} := \{T \subseteq S : |T| = k\}$. A hypergraph H is *k -uniform* if $E(H) \subseteq \binom{V(H)}{k}$, and we often refer to a k -uniform hypergraph as a *k -graph*. Let H be a k -graph and $T \subseteq V(H)$. The *neighbourhood* $N_H(T)$ of T in H denotes the family of subsets $S \subseteq V(H) \setminus T$ such that $T \cup S \in E(H)$. When $T = \{x\}$, we express $N_H(\{x\})$ by $N_H(x)$. The *degree* $d_H(T)$ of T in H is defined as $d_H(T) = |N_H(T)|$. For an integer ℓ with $1 \leq \ell \leq k - 1$, let $\delta_\ell(H) = \min\{d_H(T) : T \in \binom{V(H)}{\ell}\}$ denote the *minimum ℓ -degree* of H . For simplicity, we often write $\delta_1(H)$ as $\delta(H)$ and term it the *minimum vertex degree* of H . Throughout this paper, we often identify $E(H)$ with H when there is no confusion. A *matching* in a hypergraph H denotes a set of pairwise disjoint edges in H . A matching in H is *perfect* if it covers all vertices of H .

1.1 Background

The study of *discrepancy* of hypergraphs \mathcal{H} is a fundamental subject in combinatorics, with the aim of estimating the maximum imbalance guaranteed to occur on some edge $e \in E(\mathcal{H})$ in every 2-coloring of $V(\mathcal{H})$. For historical context, we reference Chapter 13 of [2] and Chapter 4 of [28]. A natural multicolor extension can be formally described as follows. For any integer $r \geq 2$, let \mathcal{H} be a hypergraph and $f : V(\mathcal{H}) \rightarrow [r]$ be a r -coloring of its vertices. For any color $c \in [r]$ and any edge $e \in E(\mathcal{H})$, let $c(e)$ denote the number of vertices in e colored with c under f , and let $D_f(e) = \max_{c \in [r]} \left(c(e) - \frac{|e|}{r} \right)$. The *r -color discrepancy* of a hypergraph \mathcal{H} is then defined as $D_r(\mathcal{H}) = \min_{f: V(\mathcal{H}) \rightarrow [r]} \max_{e \in E(\mathcal{H})} D_f(e)$. Therefore, the 2-color discrepancy aligns with the initial definition of the discrepancy.

Initially introduced by Erdős in the 1960s, a well-explored discrepancy problem in the context of graphs considers $V(\mathcal{H})$ as the edge set of a graph G , with $E(\mathcal{H})$ representing a collection of subgraphs of G possessing specific properties such as spanning trees, Hamilton cycles and perfect matchings.

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Two notable early results include Erdős-Spencer [9] on the discrepancy of cliques and Erdős-Füredi-Loebl-Sós [8] on the discrepancy of a given spanning tree in complete graphs. In 2020, Balogh-Csaba-Jing-Pluhár [3] revisited this problem for general graphs G , and since then, there has been extensive research investigating the r -color discrepancy of various properties of subgraphs in both graphs and hypergraphs, including [4, 5, 6, 7, 10, 11, 12, 13, 14, 15, 27] in the literature.

In these recent studies, a significant focus has been on exploring the minimum (ℓ -)degree threshold for perfect matchings with high discrepancy in graphs and hypergraphs.¹ This exploration stems from the original problem of determining the minimum ℓ -degree threshold for the existence of perfect matchings in uniform hypergraphs, a subject that has garnered considerable interest in recent decades. Formally speaking, we are given integers k, ℓ, n with $1 \leq \ell \leq k - 1$ and $n \equiv 0 \pmod{k}$. Let $m_\ell(k, n)$ denote the smallest integer m such that every n -vertex k -graph H with $\delta_\ell(H) \geq m$ contains a perfect matching. We have the following general asymptotic lower bound on $m_\ell(k, n)$, where the explicit construction of the two families of k -graphs can be found in [25, 34]:

$$\text{By setting } f^\ell(k) := \limsup_{n \rightarrow \infty} \frac{m_\ell(k, n)}{\binom{n-\ell}{k-\ell}}, \text{ it holds that } f^\ell(k) \geq \max \left\{ \frac{1}{2}, 1 - \left(1 - \frac{1}{k}\right)^{k-\ell} \right\}. \quad (1)$$

For brevity, we abbreviate $m_1(k, n)$ as $m(k, n)$ and $f^1(k)$ as $f(k)$. Note that the well-known Dirac's theorem implies that $f(2) = 1/2$ for graphs. It is conjectured (see [18, 23]) that the above lower bound should hold as an equality for all integers $k > \ell \geq 1$. This conjecture remains open in general, although significant progress has been achieved, including the confirmation of $f(3) = 5/9$ in [18, 20, 24], the confirmation of $f(4) = 1 - (3/4)^3$ in [1, 21], the confirmation of $f(5) = 1 - (4/5)^4$ in [1], the confirmation of $f^{k-1}(k) = 1/2$ in [22, 31, 32], as well as the verification of $f^\ell(k) = 1/2$ for $\ell \geq k/2$ in [29, 33] and towards better bounds $\ell \geq 0.4k$ in [16, 26]. For further discussions, we refer to [25, 30, 35].

Now we formalize the problem of determining the minimum ℓ -degree threshold for perfect matchings with high discrepancy in r -edge-colored k -uniform hypergraphs. Fix integers r, k, ℓ with $r \geq 2$ and $1 \leq \ell \leq k - 1$. Let $h_r^\ell(k)$ denote the smallest constant $h > 0$ satisfying that for any $\epsilon > 0$, there exists a constant $\delta = \delta(\epsilon, r, k, \ell) > 0$ so that the following holds: for every n -vertex k -graph H and every r -coloring of $E(H)$, where $n \equiv 0 \pmod{k}$ is sufficiently large, if $\delta_\ell(H) \geq (h + \epsilon) \cdot \binom{n-\ell}{k-\ell}$, then there exists a perfect matching of H with at least $\frac{n}{rk} + \delta \cdot n$ edges with the same color. For brevity, we will write $h_r^1(k)$ as $h_r(k)$. It is evident from the definition that

$$h_r^\ell(k) \geq f^\ell(k) \text{ holds for all } r \geq 2 \text{ and } 1 \leq \ell \leq k - 1. \quad (2)$$

Balogh, Csaba, Jing and Pluhár [3] first established the minimum degree threshold for perfect matchings with high discrepancy in graphs by showing that $h_2(2) = \frac{3}{4}$. Later, this was generalized to the multicolor version in Freschi-Hyde-Lada-Treglown [10] and Gishboliner-Krivelevich-Michaeli [14], where it was proven that $h_r(2) = \frac{r+1}{2r}$ holds for all $r \geq 2$. This bound is attained by the following r -edge-colored graph G : $V(G)$ consists of subsets V_1, \dots, V_r with $|V_i| = \frac{n}{2r}$ for all $i < r$ and $|V_r| = \frac{(r+1)n}{2r}$; the edges of G comprises all pairs intersecting with V_r , where for each $i \in [r]$ the edges between V_i and V_r are colored by color i . The same problem was also explored in random graphs in [13]. Recently, Gishboliner, Glock and Sgueglia [12] determined the minimum co-degree threshold by deriving from their main result that $h_r^{k-1}(k) = \frac{1}{2} = f^{k-1}(k)$ holds for all $r \geq 2$ and $k \geq 3$. Simultaneously and independently, Balogh, Treglown and Zárate-Guerén [5] demonstrated that $h_r^\ell(k) = f^\ell(k)$ holds for all $r \geq 2$ and $2 \leq \ell \leq k - 1$. This is remarkable given that generally the exact value of $f^\ell(k)$ remains unknown. Now we are left with the only remaining cases when $\ell = 1$, namely, the minimum vertex degree threshold for perfect matchings with high r -color discrepancy in k -graphs. This is summarized in the following problem statement.

Problem 1. *Determine $h_r(k)$ for all $k \geq 3$ and $r \geq 2$.*

¹Here and throughout, high discrepancy in a hypergraph H always means a discrepancy with linear size $\Omega(|V(H)|)$.

It is worth mentioning that there is a notable difference in the known cases between graphs and hypergraphs. In graphs, we observe that $h_r(2) > f(2)$ for all $r \geq 2$, whereas in k -uniform hypergraphs for every $k \geq 3$, all known cases indicate that $h_r^\ell(k) = f^\ell(k)$ for all $r \geq 2$, where $2 \leq \ell \leq k - 1$. As inquired by Gishboliner, Glock, and Sgueglia [12], it would be interesting to know whether there exists some case in k -uniform hypergraphs with $k \geq 3$, where the discrepancy threshold is strictly larger than the corresponding existence threshold (namely, $h_r(k) > f(k)$ in the remaining cases). In particular, Balogh, Treglown and Zárata-Guerén [5] considered the discrepancy threshold in 3-graphs. They observed $h_2(3) \geq \frac{3}{4}$ from the following 3-graph H : The vertex set $V(H)$ consists of two subsets A and B , both of size $n/2$, while $E(H)$ comprises all red-colored edges with two endpoints in A and one endpoint in B , along with all blue-colored edges having one endpoint in A and two endpoints in B . The authors of [5] (see Question 4.2) posed the question of whether $h_2(3) = \frac{3}{4}$ holds true.

1.2 Our results

In this paper, we advance and complete the aforementioned line of research by establishing the minimum vertex degree threshold of perfect matchings with high r -color discrepancy in k -graphs for all $r, k \geq 2$. To achieve this, we identify this discrepancy threshold using a novel family of k -graphs and unveil new phenomena that have not been explored in previous results.

To set the stage for our findings, we initiate our presentation by examining the lower bound with a detailed characterization of the extremal k -graphs as follows. For integers $k \geq 3$ and $r \geq 2$, let \mathbb{N}_{k-1}^r denote the set consisting of all vectors $\vec{\mathbf{a}} = (a_1, a_2, \dots, a_r) \in \mathbb{N}^r$ such that $a_1 + a_2 + \dots + a_r = k - 1$ and $0 \leq a_1 \leq a_2 \leq \dots \leq a_r$. (See Figure 1 for illustration.)

Definition 2. For any integer n divisible by rk , $\vec{\mathbf{a}} = (a_1, a_2, \dots, a_r) \in \mathbb{N}_{k-1}^r$ and $\vec{\mathbf{b}} = (b_1, b_2, \dots, b_r) \in \mathbb{N}_r^{k+1}$ such that $b_1 \geq 1$. Let $H(n, \vec{\mathbf{a}})$ denote the k -graph on vertex set $V_1 \cup V_2 \cup \dots \cup V_r$ with $|V_i| = \frac{ra_i + 1}{rk}n$ for each $i \in [r]$ and consisting of all edges e with $|V(e) \cap V_i| = a_i + 1$ and $|V(e) \cap V_j| = a_j$ for every $i \in [r]$ and all $j \in [r] \setminus \{i\}$.

For any $k \geq 3$ and $r \geq 2$, we define $\mathcal{H}_{r,k} = \{H(n, \vec{\mathbf{a}}) \mid \vec{\mathbf{a}} \in \mathbb{N}_{k-1}^r\}$ and

$$g_r(k) := \max_{\vec{\mathbf{a}} \in \mathbb{N}_{k-1}^r} \left\{ \lim_{n \rightarrow \infty} \frac{\delta(H(n, \vec{\mathbf{a}}))}{\binom{n-1}{k-1}}, \mid n \text{ is divisible by } rk \right\}.$$

It is notable that each $H(n, \vec{\mathbf{a}})$ contains exactly r types of edges: for every $i \in [r]$, let E_i denote the edge set containing all edges e with $|V(e) \cap V_i| = a_i + 1$ and $|V(e) \cap V_j| = a_j$ for each $j \in [r] \setminus \{i\}$. Let $H^*(n, \vec{\mathbf{a}})$ be the r -edge-colored k -graph obtained from $H(n, \vec{\mathbf{a}})$ by assigning color i to each edge in E_i for all $i \in [r]$. It is observed that every perfect matching in $H^*(n, \vec{\mathbf{a}})$ has exactly n/rk edges with color i for every $i \in [r]$, thereby indicating that $h_r(k) \geq g_r(k)$. Combining this with (2), we derive the following lower bound for $h_r(k)$.

Proposition 3. For any $k \geq 3$ and $r \geq 2$, it holds that $h_r(k) \geq \max\{f(k), g_r(k)\}$.

Our main result is stated below, demonstrating that the above lower bound is actually optimal.

Theorem 4. For any integers $k \geq 3$ and $r \geq 2$, there exists a real $\gamma = \gamma(k, r) > 0$ such that the following holds. For any $\eta > 0$ and any k -uniform hypergraph H with a sufficiently large number of vertices $n \geq n_0(k, r, \eta)$, where $n \equiv 0 \pmod{k}$, if

$$\delta(H) > (\max\{f(k), g_r(k)\} + \eta) \cdot \binom{n-1}{k-1}$$

then for any r -coloring of the edges of H , there is a perfect matching in H with at least $\frac{n}{rk}(1 + \gamma \cdot \eta)$ edges of the same color.

²We refer to Lemma 14 for a precise formulation of $g_r(k)$.

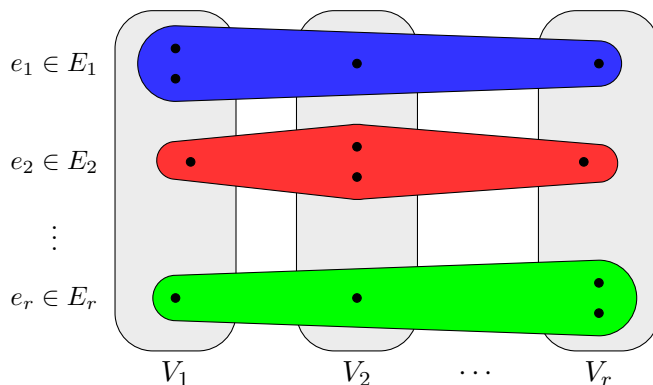


Figure 1: The $(r + 1)$ -graph $H(n, \vec{a}) \in \mathcal{H}_{r,r+1}$ for $\vec{a} = (1, 1, \dots, 1) \in \mathbb{N}_r^r$

Now combining Proposition 3 with Theorem 4, we can express the solution to Problem 1 as

$$h_r(k) = \max\{f(k), g_r(k)\} \text{ for all } k \geq 3 \text{ and } r \geq 2.$$

For any given values of k and r , this involves a finite optimization process, however determining whether $h_r(k) > f(k)$ is not a straightforward task. In the subsequent result, we further identify the unique extremal configurations for all instances, with the exception of finite specific cases, namely when $r = 2$ and $6 \leq k \leq 16$.

Theorem 5. *For any integers $k \geq 3$ and $r \geq 2$, it holds that*

$$h_r(k) = \max\{f(k), g_r(k)\},$$

where we have

- $f(k) > g_r(k)$ if one of the following holds:
 - (1) $r = 2$ and $k \geq 17$; (2) $r = 3$ and $k \geq 5$; (3) $r = 4$ and $k \geq 4$; (4) $r \geq 5$.
- $g_r(k) > f(k)$ if one of the following holds:
 - (1) $r = 2$ and $k = 3$, which is attained by the 3-graph $H(n, (1, 1))$;
 - (2) $r = 2$ and $k = 4$, which is attained by the 4-graph $H(n, (1, 2))$;
 - (3) $r = 2$ and $k = 5$, which is attained by the 5-graph $H(n, (0, 4))$;
 - (4) $r = 3$ and $k = 3$, which is attained by the 3-graph $H(n, (0, 0, 2))$;
 - (5) $r = 3$ and $k = 4$, which is attained by the 4-graph $H(n, (0, 0, 3))$;
 - (6) $r = 4$ and $k = 3$, which is attained by the 3-graph $H(n, (0, 0, 0, 2))$.

The proof of this theorem relies on established results on $f(k)$ and utilizes analytical optimization techniques on $g_r(k)$. It indicates that apart from a finite number of cases, the r -color discrepancy threshold aligns with the corresponding existence threshold for perfect matchings in k -uniform hypergraphs for $k \geq 3$ and $r \geq 2$. Conversely, notably, there exist instances where the discrepancy threshold exceeds the corresponding existence threshold, thus offering an affirmative answer to the query posed by Gishboliner, Glock, and Sgueglia [12]. Furthermore, observing that the 3-graph $H(n, (1, 1))$ coincides with the example provided by Balogh, Treglown, and Zárate-Guerén [5], this shows that $h_2(3) = \frac{3}{4}$ indeed holds, confirming the question posted in [5] for the case $k = 3$ and $r = 2$.

The rest of the paper is organized as follows. In Section 2, we introduce some necessary notations and preliminary results. In Section 3, we establish Lemma 9, offering the fundamental structure for the extremal hypergraphs we consider. In Section 4 and Section 5, we prove Theorem 4 and Theorem 5, respectively. Finally, in Section 6 we discuss some remarks and questions for further consideration.

2 Preliminaries

In this section, we introduce notations and some preliminary results that will be used in the forthcoming proof. Throughout, let $r \geq 2$ be an integer, H be a k -graph and c be an r -edge-coloring of H such that $c : E(H) \rightarrow \{C_1, C_2, \dots, C_r\}$.

We primarily adhere to the terminologies used in [5]. Given $x \in V(H)$ and $A \subseteq V(H)$, we write xA or Ax to denote the set $\{x\} \cup A$. By $H - A$, we mean the k -graph obtained from H by deleting all vertices in A . An edge of H is called a C_i -edge if it is colored by C_i in c . For a subgraph F of H , the *color profile* $\vec{cp}(F)$ of F denotes the r -tuple (x_1, x_2, \dots, x_r) , where x_i denotes the number of C_i -edges in F for each $i \in [r]$. The following definitions are crucial in the coming proofs.

Definition 6. Let $\{u, v\} \in \binom{V(H)}{2}$ and $T \in N_H(u) \cap N_H(v)$. We say uTv is

- **S**, if $c(T \cup \{u\}) = c(T \cup \{v\})$, or
- **$C_i C_j$** , if $c(T \cup \{u\}) = C_i$ and $c(T \cup \{v\}) = C_j$ where $i \neq j$.

Definition 7. Let $\{u, v\} \in \binom{V(H)}{2}$. We say that an ordered pair (u, v) is

- **type S** if there are at least $k^2 \binom{n-2}{k-2}$ sets $T \in N_H(u) \cap N_H(v)$ such that uTv is **S**, or
- **type $C_i C_j$** if there are at least $k^2 \binom{n-2}{k-2}$ sets $T \in N_H(u) \cap N_H(v)$ such that uTv is **$C_i C_j$** .

We point out that (u, v) is type **$C_i C_j$** if and only if (v, u) is type **$C_j C_i$** , and it is possible that (u, v) has multiple types. In the coming proofs, we will frequently use the following simple observations: If (u, v) is type **S** (or **$C_i C_j$**), then for any subset A of size less than k^2 , there exists a set T in $N_H(u) \cap N_H(v)$ such that $T \cap A = \emptyset$ and uTv is **S** (or **$C_i C_j$**); in particular, one can find at least k disjoint sets T_i in $N_H(u) \cap N_H(v)$ such that each $uT_i v$ is **S** (or **$C_i C_j$**).

A subgraph F of H is called *good* (with respect to a given coloring c), if F contains two perfect matchings that have different color profiles. As we will see later, it becomes evident that good subgraphs are essential in the construction of color-biased perfect matchings of H .

In what follows, we introduce three kinds of good subgraphs (referred to as *gadgets*) that will be heavily used in the proofs.

Good Gadget 1. Let $u, v \in V(H)$ be distinct vertices. A good gadget G of the first kind is a subgraph of H on $2k$ vertices such that

- $G := H[\{u, v\} \cup T_1 \cup T_2]$, where $T_1, T_2 \in N_H(u) \cap N_H(v)$ are disjoint, and
- $\{T_1 u, T_2 v\}$ and $\{T_1 v, T_2 u\}$ are two perfect matchings of G with different color profiles.

Good Gadget 2. Let $\{u_1, u_2, u_3\} \in \binom{V(H)}{3}$. A good gadget G of the second kind is a subgraph of H on $3k$ vertices such that

- $G := H[\{u_1, u_2, u_3\} \cup T_1 \cup T_2 \cup T_3]$, where $T_1 \in N_H(u_2) \cap N_H(u_3)$, $T_2 \in N_H(u_1) \cap N_H(u_3)$ and $T_3 \in N_H(u_1) \cap N_H(u_2)$ are pairwise disjoint, and
- $\{T_1 u_2, T_2 u_3, T_3 u_1\}$ and $\{T_1 u_3, T_2 u_1, T_3 u_2\}$ are two perfect matchings of G with two different color profiles.

Good Gadget 3. Let e, f be two edges of H such that $V(e) \setminus V(f) = \{u_1, \dots, u_\ell\}$ and $V(f) \setminus V(e) = \{v_1, \dots, v_\ell\}$ for some $1 \leq \ell \leq k$. A good gadget G of the third kind is a subgraph of H on $(\ell + 1)k$ vertices such that

- $G := H[V(e) \cup V(f) \cup T_1 \cup \dots \cup T_\ell]$, where $T_i \in N_H(u_i) \cap N_H(v_i)$ for all $i \in [\ell]$ are disjoint, and
- $\{u_1 T_1, \dots, u_\ell T_\ell, f\}$ and $\{v_1 T_1, \dots, v_\ell T_\ell, e\}$ are two perfect matchings of G with different color profiles.

Finally, we need the following classic result of Hilton and Milner [19]. We say that two families \mathcal{A}, \mathcal{B} of sets are *cross-intersecting*, if $A \cap B \neq \emptyset$ for every $A \in \mathcal{A}$ and every $B \in \mathcal{B}$.

Theorem 8 (Hilton and Milner [19]). *Suppose that $m > 2\ell > 0$ and $\mathcal{A}, \mathcal{B} \subseteq \binom{[m]}{\ell}$ are two non-empty and cross-intersecting families. Then we have*

$$|\mathcal{A}| + |\mathcal{B}| \leq \binom{m}{\ell} - \binom{m-\ell}{\ell} + 1.$$

3 Extremal structure and minimum degree

In this section, we prove our key technical lemma – Lemma 9, which forms the structural foundation of the main theorem.

Before introducing this lemma, we define k -graphs that extend Definition 2 and highlight a crucial observation. Let $n, k \geq 3$ and $r \geq 2$ be integers such that n is dividable by rk . For any $\vec{\mathbf{a}} = (a_1, \dots, a_r) \in \mathbb{N}_{k-1}^r$ and any r -partition $\vec{V} = (V_1, \dots, V_r)$ of n vertices, let $H(\vec{V}, \vec{\mathbf{a}})$ denote the k -graph with vertex set $V_1 \cup V_2 \cup \dots \cup V_r$ consisting of all edges e with $|V(e) \cap V_i| = a_i + 1$ and $|V(e) \cap V_j| = a_j$ for all $1 \leq i \neq j \leq r$. It is not hard to see that $H(\vec{V}, \vec{\mathbf{a}})$ contains a perfect matching if and only if $|V_i| \geq \frac{a_i}{k}n$ holds for every $i \in [r]$. Let $H^*(\vec{V}, \vec{\mathbf{a}})$ be the r -edge-colored k -graph obtained from $H(\vec{V}, \vec{\mathbf{a}})$ by coloring each edge e , satisfying $|V(e) \cap V_i| = a_i + 1$, with the color C_i for every $i \in [r]$. For a subgraph F , we write $C_i(F)$ as the number of C_i -edges contained in F . Then one can observe that for every perfect matching \mathcal{M} in $H^*(\vec{V}, \vec{\mathbf{a}})$ and for every $i \in [r]$,

$$\text{it holds that } |V_i| = a_i \frac{n}{k} + C_i(\mathcal{M}). \quad (3)$$

There is still another type of k -graphs such that there does not exist any kind of good gadgets for some r -coloring. Indeed, it has similar structure as $H(n, \vec{\mathbf{a}})$. Let $n, k \geq 3$ and $r \geq 2$ be integers such that n is dividable by rk . For any $\vec{\mathbf{b}} \in \mathbb{N}_{k+1}^r$ such that $b_1 \geq 1$, let $\tilde{H}(n, \vec{\mathbf{b}})$ denote the k -graph on vertex set $V_1 \cup V_2 \cup \dots \cup V_r$ with $|V_i| = \frac{rb_i-1}{rk}n$ for each $i \in [r]$ and consisting of all edges e with $|V(e) \cap V_i| = b_i - 1$ and $|V(e) \cap V_j| = b_j$ for every $i \in [r]$ and all $j \in [r] \setminus \{i\}$. Let $\tilde{H}^*(n, \vec{\mathbf{b}})$ be the r -edge-colored k -graph obtained from $\tilde{H}(n, \vec{\mathbf{b}})$ by assigning color i to each edge in E_i for all $i \in [r]$, where E_i consists of all edges e in $\tilde{H}(n, \vec{\mathbf{b}})$ such that $|V(e) \cap V_i| = b_i - 1$ and $|V(e) \cap V_j| = b_j$ for each $j \in [r] \setminus \{i\}$. It is easy to check that every perfect matching of $\tilde{H}^*(n, \vec{\mathbf{b}})$ has exactly n/rk edges with color i for every $i \in [r]$.

Under the same setting of n, k and r , for any $\vec{\mathbf{b}} = (b_1, \dots, b_r) \in \mathbb{N}_{k+1}^r$ satisfying $b_1 \geq 1$ and any r -partition $\vec{V} = (V_1, \dots, V_r)$ of n vertices, let $\tilde{H}(\vec{V}, \vec{\mathbf{b}})$ denote the k -graph with vertex set $V_1 \cup V_2 \cup \dots \cup V_r$ consisting of all edges e with $|V(e) \cap V_i| = b_i - 1$ and $|V(e) \cap V_j| = b_j$ for all $1 \leq i \neq j \leq r$. Let $\tilde{H}^*(\vec{V}, \vec{\mathbf{b}})$ be the r -edge-colored k -graph obtained from $\tilde{H}(\vec{V}, \vec{\mathbf{b}})$ by coloring each edge e , satisfying $|V(e) \cap V_i| = b_i - 1$, with the color C_i for every $i \in [r]$. Then one can observe that for every perfect matching \mathcal{M} in $\tilde{H}^*(\vec{V}, \vec{\mathbf{b}})$ and for every $i \in [r]$,

$$\text{it holds that } |V_i| = b_i \frac{n}{k} - C_i(\mathcal{M}). \quad (4)$$

Our key lemma reads as follows. It establishes the necessity of the presence of $H^*(\vec{V}, \vec{\mathbf{a}})$ or $\tilde{H}^*(\vec{V}, \vec{\mathbf{b}})$ for some specific \vec{V} and $\vec{\mathbf{a}} \in \mathbb{N}_{k-1}^r$ or $\vec{\mathbf{b}} \in \mathbb{N}_{k+1}^r$ satisfying $b_1 \geq 1$ in determining the extremal structure of k -graphs.

Lemma 9. *Let n, k, r be positive integers with $n \geq 2k^2$ and $r \geq 2$. Let H be an r -edge-colored n -vertex k -graph with the edge-coloring $c : E(H) \rightarrow \{C_1, C_2, \dots, C_r\}$ such that H has at least one C_i -edge for each $i \in [r]$. Suppose the following conditions hold:*

- $\delta(H) > \frac{1}{2} \binom{n-1}{k-1} + \frac{k^2+1}{2} \binom{n-2}{k-2}$, and
- H does not contain any good gadgets of any kind.

Then by possibly renaming the colors, there exists an r -partition \vec{V} of $V(H)$ and a vector $\vec{\mathbf{a}} \in \mathbb{N}_{k-1}^r$ or a vector $\vec{\mathbf{b}} \in \mathbb{N}_{k+1}^r$ satisfying $b_1 \geq 1$ such that H is an r -edge-colored subgraph of $H^*(\vec{V}, \vec{\mathbf{a}})$ or $\tilde{H}^*(\vec{V}, \vec{\mathbf{b}})$.

In the remainder of this section, we start by establishing several lemmas that describe the local structures of k -graphs with high minimum degrees but without any good gadgets. Following that, we present the proof of Lemma 9 in the next subsection. Finally, we conclude this section with a lemma that describes the minimum vertex degree of the extremal k -graphs under consideration.

3.1 Local structures

Now, we proceed to prove a series of three lemmas, with each lemma building upon the previous one. These lemmas are aimed at describing the local structure of hypergraphs using the terminologies provided in Definitions 6 and 7.

Lemma 10. *Let n, k, r be positive integers with $n > 2k$ and $r \geq 2$. Let H be an n -vertex k -graph and let $c : E(H) \rightarrow \{C_1, C_2, \dots, C_r\}$. Assume that H does not contain any good gadgets of any kind. Let $\{u, v\} \in \binom{V(H)}{2}$ satisfy*

$$|N_H(u) \cap N_H(v)| > \binom{n-2}{k-1} - \binom{(n-2)-(k-1)}{k-1} + 1. \quad (5)$$

Then there exists $\mathbf{U} = \mathbf{S}$ or $\mathbf{C}_i\mathbf{C}_j$ for some $i \neq j$ such that uTv is \mathbf{U} for all $T \in N_H(u) \cap N_H(v)$.

Proof. We first claim that if there exists $T_1 \in N_H(u) \cap N_H(v)$ such that uT_1v is \mathbf{S} , then for all $T \in N_H(u) \cap N_H(v)$, uTv is \mathbf{S} . Suppose this is not the case. Then $\mathcal{A} := \{T \in N_H(u) \cap N_H(v) : uTv \text{ is } \mathbf{S}\}$ and $\mathcal{B} := \{T \in N_H(u) \cap N_H(v) : uTv \text{ is not } \mathbf{S}\}$ both are non-empty families contained in $\binom{V(H) \setminus \{u, v\}}{k-1}$. So $|\mathcal{A}| + |\mathcal{B}| = |N_H(u) \cap N_H(v)|$ satisfies (5). By Lemma 8, there exist two disjoint $(k-1)$ -subsets $R_1, R_2 \in N_H(u) \cap N_H(v)$ such that uR_1v is \mathbf{S} and uR_2v is not \mathbf{S} . So uR_2v is $\mathbf{C}_i\mathbf{C}_j$ for some $1 \leq i \neq j \leq r$. Then $\{uR_1, vR_2\}$ and $\{vR_1, uR_2\}$ are two perfect matchings of $H[R_1 \cup R_2 \cup \{u, v\}]$ with different color profiles. This gives a good gadget of the first kind, a contradiction.

By the this claim, we may assume that for every $T \in N_H(u) \cap N_H(v)$, uTv is not \mathbf{S} . We need to show they must belong to the same kind, say $\mathbf{C}_i\mathbf{C}_j$ for some $i \neq j$. Suppose for a contradiction that this is not the case. Then using Lemma 8 (and the argument in the previous paragraph) again, we see that there exist two disjoint $(k-1)$ -subsets $R_1, R_2 \in N_H(u) \cap N_H(v)$ such that uR_1v is $\mathbf{C}_i\mathbf{C}_j$ and uR_2v is not $\mathbf{C}_i\mathbf{C}_j$. We may assume that uR_2v is $\mathbf{C}_\ell\mathbf{C}_m$, where either $i \neq \ell$ or $j \neq m$. It is evident to see that in either case, $\{uR_1, vR_2\}$ and $\{vR_1, uR_2\}$ are two perfect matchings of $H[T_1 \cup T_2 \cup \{u, v\}]$ with different color profiles, again a contradiction. This completes the proof. \square

The subsequent lemma shows that under some mild assumptions on hypergraphs, each ordered pair of vertices possesses a unique type.

Lemma 11. *Let n, k, r be integers with $n \geq 2k^2$ and $r \geq 2$. Let H be an n -vertex k -graph and let $c : E(H) \rightarrow \{C_1, C_2, \dots, C_r\}$. Assume that H does not contain any good gadgets of any kind and*

$$\delta(H) > \frac{1}{2} \binom{n-1}{k-1} + \frac{k^2+1}{2} \binom{n-2}{k-2}. \quad (6)$$

Then the following hold:

- (A). Every ordered pair (u, v) of $V(H)$ has a unique type, which is either type \mathbf{S} or type $\mathbf{C}_i\mathbf{C}_j$ for some $1 \leq i \neq j \leq r$;
- (B). If c is not monochromatic, then there exists $\{u, v\} \in \binom{V(H)}{2}$ such that (u, v) is not type \mathbf{S} .

Proof. For (A), consider any $\{u, v\} \in \binom{V(H)}{2}$. Since $N_H(u) \cap N_H(v)$ is contained in $\binom{V(H) \setminus \{u, v\}}{k-1}$, using (6) we can derive that $|N_H(u) \cap N_H(v)|$ is at least

$$\begin{aligned} & |N_H(u) - \{v\}| + |N_H(v) - \{u\}| - \binom{n-2}{k-1} \geq 2 \left(\delta(H) - \binom{n-2}{k-2} \right) - \binom{n-2}{k-1} \\ & > k^2 \binom{n-2}{k-2} > \sum_{i=1}^{k-1} \binom{n-i-2}{k-2} + 1 = \binom{n-2}{k-1} - \binom{(n-2) - (k-1)}{k-1} + 1. \end{aligned}$$

That is, $|N_H(u) \cap N_H(v)| > k^2 \binom{n-2}{k-2}$ and it satisfies (5). By Lemma 10, there exists $\mathbf{U} = \mathbf{S}$ or $\mathbf{C}_i \mathbf{C}_j$ for some $i \neq j$ such that uTv is \mathbf{U} for all $T \in N_H(u) \cap N_H(v)$. Hence, (u, v) has a unique type, which is either type \mathbf{S} or type $\mathbf{C}_i \mathbf{C}_j$.

To show (B), suppose for a contradiction that any pair (u, v) of $V(H)$ is type \mathbf{S} . Assume that H contains a C_1 -edge. Let \mathcal{A} be the family consisting of all C_1 -edges in H and $\mathcal{B} = E(H) \setminus \mathcal{A}$. Since c is not monochromatic, \mathcal{A} and \mathcal{B} are non-empty. By (6) and since $n \geq 2k^2$, it is easy to see that

$$|\mathcal{A}| + |\mathcal{B}| = e(H) > \left(\frac{1}{2} \binom{n-1}{k-1} + 1 \right) \frac{n}{k} \geq k \binom{n-1}{k-1} + 2k \geq \binom{n}{k} - \binom{n-k}{k} + 2.$$

By Lemma 8, there exist two disjoint edges e, f in H such that e is a C_1 -edge and f is not. Write $V(e) = \{u_1, \dots, u_k\}$ and $V(f) = \{v_1, \dots, v_k\}$. Since every (u_i, v_i) is type \mathbf{S} for $i \in [k]$, by the observations after Definition 7, there exist $T_i \in N_H(u_i) \cap N_H(v_i)$ for $i \in [k]$ which are pairwise disjoint. Then $\{u_1 T_1, \dots, u_k T_k, f\}$ and $\{v_1 T_1, \dots, v_k T_k, e\}$ are two perfect matchings of $H_0 = H[V(e) \cup V(f) \cup (\bigcup_{i=1}^k T_i)]$ with different color profiles, which shows that H_0 is a good gadget of the third kind, a contradiction. This finishes the proof. \square

The final lemma in this subsection characterizes all possible pairwise type situations between any three vertices.

Lemma 12. *Let n, k, r be positive integers with $n \geq 2k^2$ and $r \geq 2$. Let H be an n -vertex k -graph and let $c : E(H) \rightarrow \{C_1, C_2, \dots, C_r\}$. Assume that H satisfies (6) and does not contain any good gadgets of any kind. Then the following hold for any $\{u_1, u_2, u_3\} \in \binom{V(H)}{3}$.*

- (A). *If (u_1, u_2) is type \mathbf{S} and (u_1, u_3) is type $\mathbf{C}_i \mathbf{C}_j$, then (u_2, u_3) is also type $\mathbf{C}_i \mathbf{C}_j$;*
(B). *If every (u_i, u_j) is not type \mathbf{S} for $1 \leq i < j \leq 3$, then there exist distinct $\alpha, \beta, \gamma \in [r]$ such that one of the following happens:*

- (a) *(u_1, u_2) is type $\mathbf{C}_\alpha \mathbf{C}_\beta$, (u_2, u_3) is type $\mathbf{C}_\beta \mathbf{C}_\gamma$, and (u_1, u_3) is type $\mathbf{C}_\alpha \mathbf{C}_\gamma$;*
(b) *(u_1, u_2) is type $\mathbf{C}_\beta \mathbf{C}_\alpha$, (u_2, u_3) is type $\mathbf{C}_\gamma \mathbf{C}_\beta$, and (u_1, u_3) is type $\mathbf{C}_\gamma \mathbf{C}_\alpha$;*

Proof. For (A), suppose that (u_2, u_3) is not type $\mathbf{C}_i \mathbf{C}_j$. By Lemma 11, (u_2, u_3) is type \mathbf{S} or $\mathbf{C}_\ell \mathbf{C}_m$ where $\ell \neq i$ or $m \neq j$. Then there exist three disjoint sets $T_3 \in N_H(u_1) \cap N_H(u_2)$, $T_2 \in N_H(u_1) \cap N_H(u_3)$, and $T_1 \in N_H(u_2) \cap N_H(u_3)$ such that

$$u_1 T_3 u_2 \text{ is } \mathbf{S}, \quad u_1 T_2 u_3 \text{ is } \mathbf{C}_i \mathbf{C}_j, \quad \text{and } u_2 T_1 u_3 \text{ is either } \mathbf{S} \text{ or } \mathbf{C}_\ell \mathbf{C}_m.$$

We claim that in either case, $\{u_1 T_3, u_2 T_1, u_3 T_2\}$ and $\{u_2 T_3, u_3 T_1, u_1 T_2\}$ are two perfect matchings of $H_1 = H[\{u_1, u_2, u_3\} \cup T_1 \cup T_2 \cup T_3]$ with two different color profiles. This is easy to see when $u_2 T_1 u_3$ is \mathbf{S} . Now consider when $u_2 T_1 u_3$ is $\mathbf{C}_\ell \mathbf{C}_m$. By symmetry between ℓ and m , we may assume that $\ell \neq i$. In this case, we have $c(u_1 T_3) = c(u_2 T_3)$, and $c(u_2 T_1) = \ell \notin \{m, i\} = \{c(u_3 T_1), c(u_1 T_2)\}$, confirming the claim. Hence H_1 is a good gadget of the second kind, a contradiction.

Finally we consider (B). Suppose that (u_1, u_2) is type $\mathbf{C}_i \mathbf{C}_j$, (u_1, u_3) is type $\mathbf{C}_\ell \mathbf{C}_m$ and (u_2, u_3) is type $\mathbf{C}_g \mathbf{C}_h$. Then we may choose three disjoint sets $T_3 \in N_H(u_1) \cap N_H(u_2)$, $T_2 \in N_H(u_1) \cap N_H(u_3)$ and $T_1 \in N_H(u_2) \cap N_H(u_3)$ such that

$$u_1 T_3 u_2 \text{ is } \mathbf{C}_i \mathbf{C}_j, \quad u_1 T_2 u_3 \text{ is } \mathbf{C}_\ell \mathbf{C}_m, \quad \text{and } u_2 T_1 u_3 \text{ is } \mathbf{C}_g \mathbf{C}_h.$$

Consider the multi-set $L = \{c(u_1T_3), c(u_1T_2), c(u_2T_1), c(u_2T_3), c(u_3T_1), c(u_3T_2)\}$. If there exists one color appearing in L once or three times, then clearly, $\{u_1T_3, u_2T_1, u_3T_2\}$ and $\{u_2T_3, u_3T_1, u_1T_2\}$ are two perfect matchings of $H[\{u_1, u_2, u_3\} \cup T_1 \cup T_2 \cup T_3]$ with different color profiles, a contradiction. If a color appears at least four times, then one of (u_1, u_2) , (u_1, u_3) and (u_2, u_3) must be type **S**, a contradiction. So every color must appear exactly two times in L and thus L contains exactly three distinct colors. Furthermore, for every color appears in L , it must appear in $\{c(u_1T_3), c(u_2T_1), c(u_3T_2)\}$ and $\{c(u_2T_3), c(u_3T_1), c(u_1T_2)\}$ once respectively. By definition, we have $(c(u_1T_3), c(u_2T_1), c(u_3T_2)) = (\mathbf{C}_i, \mathbf{C}_g, \mathbf{C}_m)$ and $(c(u_2T_3), c(u_3T_1), c(u_1T_2)) = (\mathbf{C}_j, \mathbf{C}_h, \mathbf{C}_\ell)$, then

$$\{\mathbf{C}_i, \mathbf{C}_g, \mathbf{C}_m\} = \{\mathbf{C}_j, \mathbf{C}_h, \mathbf{C}_\ell\}, \quad (7)$$

implying that $\mathbf{C}_i = \mathbf{C}_j$ or $\mathbf{C}_i = \mathbf{C}_h$ or $\mathbf{C}_i = \mathbf{C}_\ell$. Since (u_1, u_2) is not type **S**, then $\mathbf{C}_i \neq \mathbf{C}_j$. So either $\mathbf{C}_i = \mathbf{C}_\ell$ or $\mathbf{C}_i = \mathbf{C}_h$. If $\mathbf{C}_i = \mathbf{C}_\ell$, then suppose (u_1, u_2) is type $\mathbf{C}_\alpha\mathbf{C}_\beta$, (u_1, u_3) is type $\mathbf{C}_\alpha\mathbf{C}_\gamma$. Since (u_1, u_3) , (u_2, u_3) are not type **S**, we have α, β, γ are distinct. By (7), we have (u_2, u_3) is type $\mathbf{C}_\beta\mathbf{C}_\gamma$, which means item (a) happens. Otherwise we have $\mathbf{C}_i = \mathbf{C}_h$. In this case, suppose that (u_1, u_2) is type $\mathbf{C}_\beta\mathbf{C}_\alpha$, (u_2, u_3) is type $\mathbf{C}_\gamma\mathbf{C}_\beta$. Since (u_1, u_3) , (u_2, u_3) are not type **S**, we have α, β, γ are distinct. By (7), we have (u_1, u_3) is type $\mathbf{C}_\gamma\mathbf{C}_\alpha$, which means item (b) happens. This completes the proof of Lemma 13. \square

3.2 Proof of Lemma 9

In this subsection, we prove Lemma 9. It suffices to establish the following technically equivalent lemma. Under reasonable assumptions, it not only yields a useful partition of the vertex set of k -graphs but also provides crucial information regarding the precise locations and colors of all edges.

Lemma 13. *Let n, k, r be positive integers with $n \geq 2k^2$ and $r \geq 2$. Let H be an n -vertex k -graph and let $c : E(H) \rightarrow \{C_1, C_2, \dots, C_r\}$ such that H has at least one C_i -edge for all $i \in [r]$. Assume that H does not contain any good gadgets of any kind and*

$$\delta(H) > \frac{1}{2} \binom{n-1}{k-1} + \frac{k^2+1}{2} \binom{n-2}{k-2}.$$

Then by possibly renaming the colors, there exist a partition $V(H) = V_1 \cup V_2 \cup \dots \cup V_r$ such that the following hold.

(A). *For all $i \in [r]$, every $(u, v) \in V_i \times V_i$ is type **S**;*

(B). *One of the following holds:*

(B1) *For all $1 \leq i < j \leq r$, every $(u, v) \in V_i \times V_j$ is type $\mathbf{C}_i\mathbf{C}_j$;*

(B2) *For all $1 \leq i < j \leq r$, every $(u, v) \in V_i \times V_j$ is type $\mathbf{C}_j\mathbf{C}_i$; and*

(C). *If item (B1) holds, then there exists non-negative integers a_1, a_2, \dots, a_r with $\sum_{i=1}^r a_i = k-1$ such that for all $i \in [r]$, every C_i -edge e of H satisfies that $|V(e) \cap V_i| = a_i + 1$ and $|V(e) \cap V_j| = a_j$ for every $j \in [r] \setminus \{i\}$.*

If item (B2) holds, then there exists positive integers b_1, b_2, \dots, b_r with $\sum_{i=1}^r b_i = k+1$ such that for all $i \in [r]$, every C_i -edge e of H satisfies that $|V(e) \cap V_i| = b_i - 1$ and $|V(e) \cap V_j| = b_j$ for every $j \in [r] \setminus \{i\}$.

Proof. Consider H and c as given in the lemma. It is convenient for us to define an auxiliary graph G_H , where it has the vertex set $V(H)$ and the edge set

$$E(G_H) = \{uv \mid (u, v) \text{ is not type } \mathbf{S}\}.$$

By Lemma 11 (B), we see that $E(G_H) \neq \emptyset$. We first claim that G_H is connected. Suppose not. Then there exist two connected components say D_1, D_2 of G_H such that D_1 contains an edge say u_1u_2 .

Let v be a vertex in D_2 . By definition, (u_1, u_2) is not type **S** and (u_i, v) is type **S** for each $i = 1, 2$. However, this contradicts Lemma 12 (A), completing the proof of the claim.

For any $\{v_1, v_2, v_3\} \in \binom{V(H)}{3}$, if (v_1, v_2) and (v_1, v_3) are type **S**, then by Lemma 12 (A) again, we see that (v_2, v_3) must be type **S** as well. By repeatedly utilizing this fact, it can be deduced that the complement of G_H comprises vertex-disjoint cliques. In other words, G_H can be represented as a complete multipartite graph, say with parts V_1, \dots, V_s .

We first consider $s \geq 3$. Suppose that there are fixed vertices $u_i \in V_i$ for each $i \in [s]$. By applying Lemma 12 (B), by possibly renaming the colors, one of the following happens:

- (1) (u_1, u_2) is type $\mathbf{C}_1\mathbf{C}_2$, (u_1, u_3) is type $\mathbf{C}_1\mathbf{C}_3$;
- (2) (u_1, u_2) is type $\mathbf{C}_2\mathbf{C}_1$, (u_1, u_3) is type $\mathbf{C}_3\mathbf{C}_1$;

Suppose item (1) happens. In this case, for $4 \leq j \leq s$, since (u_1, u_2) , (u_1, u_j) and (u_2, u_j) are not type **S**. Then apply Lemma 12 (B) again, (u_1, u_j) is type $\mathbf{C}_1\mathbf{C}_\alpha$ or $\mathbf{C}_\beta\mathbf{C}_2$ for some $\alpha, \beta \in \{3, 4, \dots, r\}$. But if (u_1, u_j) is type $\mathbf{C}_\beta\mathbf{C}_2$, then (u_1, u_3) , (u_1, u_j) and (u_3, u_j) are not type **S** and do not satisfy any case of Lemma 12 (B), a contradiction. So (u_1, u_j) is type $\mathbf{C}_1\mathbf{C}_\alpha$.

Now let $2 \leq t \leq s$ be the largest integer such that we can rename the colors letting (u_1, u_j) be type $\mathbf{C}_1\mathbf{C}_j$ for $2 \leq j \leq t$. Since item (1) happens, we have $t \geq 3$. If $t < s$, then after possibly renaming the colors, we have (u_1, u_j) is type $\mathbf{C}_1\mathbf{C}_j$ for $2 \leq j \leq t$. Suppose that (u_1, u_{t+1}) is type $\mathbf{C}_1\mathbf{C}_\alpha$. By Lemma 12 (B), since (u_1, u_j) , (u_1, u_{t+1}) and (u_j, u_{t+1}) are not type **S** for $2 \leq j \leq t$, we have $\alpha \neq j$. So $\alpha \notin \{1, 2, \dots, t\}$, we can rename color α to $t+1$ so that (u_1, u_{t+1}) is type $\mathbf{C}_1\mathbf{C}_{t+1}$, a contradiction. Hence $t = s$, implying that by possibly renaming the colors, we have for every $2 \leq j \leq s$, (u_1, u_j) must be type $\mathbf{C}_1\mathbf{C}_j$. This implies that for every $1 \leq i < j \leq s$, (u_i, u_j) is type $\mathbf{C}_i\mathbf{C}_j$. In particular, this implies that $s \leq r$. Moreover, we can utilize Lemma 12 (B) to derive that

$$\text{every pair } (v_i, v_j) \in V_i \times V_j \text{ is type } \mathbf{C}_i\mathbf{C}_j \text{ for all } 1 \leq i \neq j \leq s. \quad (8)$$

If item (2) happens, we can derive the following analogously:

$$\text{every pair } (v_i, v_j) \in V_i \times V_j \text{ is type } \mathbf{C}_j\mathbf{C}_i \text{ for all } 1 \leq i \neq j \leq s. \quad (9)$$

If $s = 2$, it is no difference between items (8) and (9) and we can utilize Lemma 12 (B) to prove the same statement.

To complete the proof, it remains to show $s = r$ and the item (C). For an edge $e \in E(H)$, let $\vec{\#}(e) = (t_1, \dots, t_s, 0, \dots, 0)$ be the r -tuple, where each $t_j = |V(e) \cap V_j|$ denotes the number of vertices of $V(e)$ contained in the part V_j for $j \in [s]$. Note that $\sum_{j=1}^s t_j = k$. If e is a C_i -edge for some $i \in [r]$, we define $\vec{1}(e)$ as the r -tuple with its i^{th} entry being 1 and all other entries being 0.

Suppose that (8) happens, we claim that $\vec{\#}(e) - \vec{1}(e)$ is an invariant, denoted as $\vec{a} = (a_1, \dots, a_r)$, for all edges e in H . It suffices to show that for any two edges e, f in H , it holds that $\vec{\#}(e) + \vec{1}(f) = \vec{\#}(f) + \vec{1}(e)$. For any given $e, f \in E(H)$, there exists $\ell \in [k]$ such that $V(e) \setminus V(f) = \{u_1, \dots, u_\ell\}$ and $V(f) \setminus V(e) = \{v_1, \dots, v_\ell\}$. Using similar arguments as before (i.e., Lemma 11 (A)), there exist $T_i \in N_H(u_i) \cap N_H(v_i)$ for $1 \leq i \leq \ell$ which are pairwise disjoint. Then $M_1 = \{u_1T_1, \dots, u_\ell T_\ell, f\}$ and $M_2 = \{v_1T_1, \dots, v_\ell T_\ell, e\}$ are two perfect matchings of $H_0 = H[V(e) \cup V(f) \cup (\bigcup_{i=1}^\ell T_i)]$. Let \vec{c}_i be the color profile of the matching M_i for $i = 1, 2$. We may assume that $\vec{c}_1 = \vec{c}_2$ (as otherwise H_0 is a good gadget of the third kind). It also follows that $\vec{c}_1 - \vec{1}(f)$ denotes the color profile of $\{u_1T_1, \dots, u_\ell T_\ell\}$, while $\vec{c}_2 - \vec{1}(e)$ denotes the color profile of $\{v_1T_1, \dots, v_\ell T_\ell\}$. Note that by (8), the type $u_iT_iv_i$ only depends on the parts that contain u_i and v_i ; moreover, the common vertices in $V(e) \cap V(f)$ contribute equally to both $\vec{\#}(e)$ and $\vec{\#}(f)$. So it can be deduced that the difference between the color profiles of $\{u_1T_1, \dots, u_\ell T_\ell\}$ and $\{v_1T_1, \dots, v_\ell T_\ell\}$ equals $\vec{\#}(e) - \vec{\#}(f)$. This says that

$$(\vec{c}_1 - \vec{1}(f)) - (\vec{c}_2 - \vec{1}(e)) = \vec{\#}(e) - \vec{\#}(f),$$

or equivalently,

$$\vec{0} = \vec{c}_1 - \vec{c}_2 = (\vec{\#}(e) + \vec{1}(f)) - (\vec{\#}(f) + \vec{1}(e)).$$

This completes the proof of this claim.

Now we take two vertices $v_1 \in V_1$ and $v_2 \in V_2$. Then (v_1, v_2) is type $\mathbf{C}_1\mathbf{C}_2$. So there exists some $T \in N_H(v_1) \cap N_H(v_2)$ such that v_1Tv_2 is $\mathbf{C}_1\mathbf{C}_2$. That says, v_1T is a C_1 -edge and thus $\vec{1}(v_1T) = (1, 0, \dots, 0)$. Applying the previous claim on the edge v_1T , we see that

$$(a_1, \dots, a_r) = \vec{a} = \vec{\#}(v_1T) - \vec{1}(v_1T),$$

where the first entry of $\vec{\#}(v_1T)$ is at least one as $v_1 \in V_1$. Therefore, all entries a_i for $i \in [r]$ in \vec{a} are non-negative and satisfy that $\sum_{i=1}^r a_i = k - 1$.

We also can conclude that for all edges $e \in E(H)$, $\vec{\#}(e)$ only depends on the color of e . Since there are exactly r colors appearing, we will have r distinct forms for $\vec{\#}(e)$, each differing at one entry. This shows that $s = r$. Finally, each C_i -edge e in H satisfies $\vec{\#}(e) = \vec{a} + \vec{1}(e)$, implying that

$$|V(e) \cap V_i| = a_i + 1 \text{ and } |V(e) \cap V_j| = a_j \text{ for every } j \in [r] \setminus \{i\}.$$

Analogously, if (9) happens, then $s = r$, and there exists integers $b_1, \dots, b_r \geq 1$ such that $\sum_{i=1}^r b_i = k + 1$ and

$$|V(e) \cap V_i| = b_i - 1 \text{ and } |V(e) \cap V_j| = b_j \text{ for every } j \in [r] \setminus \{i\}.$$

This completes the proof of Lemma 12. \square

Now using the item (C) of Lemma 13, we can rapidly derive the validity of Lemma 9.

3.3 Minimum degree given by extremal structures

In this subsection, we provide precise estimations for the minimum vertex degrees of the k -graphs $H(n, \vec{a})$, $H(\vec{V}, \vec{a})$, $\tilde{H}(n, \vec{b})$ and $\tilde{H}^*(\vec{V}, \vec{b})$. For integers $x_i \geq 0$ and $m = \sum_{i=1}^r x_i$, we define $\binom{m}{x_1, \dots, x_r} := \frac{m!}{x_1! \dots x_r!}$.

Lemma 14. *Let $k \geq 3$ and $r \geq 2$ be integers. For any $\vec{a} = (a_1, \dots, a_r) \in \mathbb{N}_{k-1}^r$, the following hold.*

(A). *We have $\delta(H(n, \vec{a})) = (1 + o(1)) \cdot \binom{n-1}{k-1} \cdot g_r(\vec{a})$, where $o(1) \rightarrow 0$ as $n \rightarrow \infty$,*

$$g_r(\vec{a}) := \binom{k-1}{a_1, \dots, a_r} \cdot \prod_{i=1}^r \left(\frac{ra_i + 1}{rk} \right)^{a_i} \cdot \left(1 + \frac{a_1}{ra_1 + 1} \sum_{i=2}^r \frac{ra_i + 1}{a_i + 1} \right).$$

In particular, this implies that

$$g_r(k) = \max_{\vec{a} \in \mathbb{N}_{k-1}^r} g_r(\vec{a}).$$

(B). *Let $\vec{V} = (V_1, V_2, \dots, V_r)$ be a partition of n vertices such that $\frac{|V_i|}{n} = (1 + \epsilon_i) \frac{ra_i + 1}{rk}$. Then*

$$\text{by setting } \epsilon = \max_{1 \leq i \leq r} |\epsilon_i| \leq \frac{1}{2k}, \text{ we have } \delta(H(\vec{V}, \vec{a})) \leq (1 + 4k\epsilon + o(1)) \cdot \binom{n-1}{k-1} \cdot g_r(\vec{a}).$$

Proof. In this proof, we denote H as $H(n, \vec{a})$, $H(\vec{V}, \vec{a})$, and we define $V_1 \cup V_2 \cup \dots \cup V_r$ as the default r -partition of H . By definition, for any $e \in E(H)$, there exists some $i \in [r]$ such that $|e \cap V_i| = a_i + 1$ and $|e \cap V_j| = a_j$ for all $j \in [r] \setminus \{i\}$. Let $E_i(H)$ be the set of all edges e in H with $|e \cap V_i| = a_i + 1$.

³Recall that in this definition, we assume that $a_1 + a_2 + \dots + a_r = k - 1$ and $0 \leq a_1 \leq a_2 \leq \dots \leq a_r$.

Let $n_i = |V_i|$ for $i \in [r]$. Suppose that each $n_i \rightarrow \infty$ as $n \rightarrow \infty$. Then in both cases, we have

$$\begin{aligned} \delta(H) &= \min_{v \in V(H)} \sum_{\ell=1}^r \sum_{e \in E_\ell(H)} 1_{\{v \in e\}} \\ &= \min_{j \in [r]} \sum_{\ell=1; \ell \neq j}^r \left(\prod_{i=1; i \neq j, \ell}^r \binom{n_i}{a_i} \right) \binom{n_\ell}{a_\ell + 1} \binom{n_j - 1}{a_j - 1} + \left(\prod_{i=1; i \neq j}^r \binom{n_i}{a_i} \right) \binom{n_j - 1}{a_j} \\ &= \min_{j \in [r]} \left(\prod_{i=1; i \neq j}^r \binom{n_i}{a_i} \right) \cdot \binom{n_j - 1}{a_j} \left(1 + \frac{a_j}{n_j - a_j} \sum_{\ell=1; \ell \neq j}^r \frac{n_\ell - a_\ell}{a_\ell + 1} \right), \end{aligned}$$

thus implying that (here $o(1)$ denotes that $o(1) \rightarrow 0$ as $n \rightarrow \infty$)

$$\frac{\delta(H)}{\binom{n-1}{k-1}} = (1 + o(1)) \cdot \min_{j \in [r]} \binom{k-1}{a_1, a_2, \dots, a_r} \prod_{i=1}^r \left(\frac{n_i}{n} \right)^{a_i} \cdot \left(1 + \sum_{i=1; i \neq j}^r \frac{n_i a_j}{n_j (a_i + 1)} \right). \quad (10)$$

We first prove (A). In this case, n is divisible by rk and $n_i = \frac{ra_i+1}{rk}n$. Hence, we derive from (10)

$$\frac{\delta(H)}{\binom{n-1}{k-1}} = (1 + o(1)) \cdot \min_{j \in [r]} \binom{k-1}{a_1, a_2, \dots, a_r} \prod_{i=1}^r \left(\frac{ra_i+1}{rk} \right)^{a_i} \cdot \left(1 + \frac{a_j}{ra_j+1} \sum_{i=1; i \neq j}^r \frac{ra_i+1}{a_i+1} \right).$$

Recall that $0 \leq a_1 \leq a_2 \leq \dots \leq a_r$. To prove (A), it remains to show that for all $2 \leq j \leq r$,

$$\frac{a_1}{ra_1+1} \sum_{i=2}^r \frac{ra_i+1}{a_i+1} \leq \frac{a_j}{ra_j+1} \sum_{i=1; i \neq j}^r \frac{ra_i+1}{a_i+1} \quad (11)$$

Observing that for $0 \leq a \leq b$, it holds that $\frac{a}{ra+1} \leq \frac{b}{rb+1}$ and $\frac{ra+1}{a+1} \leq \frac{rb+1}{b+1}$, we can deduce that

$$\frac{a_1}{ra_1+1} \sum_{i=2; i \neq j}^r \left(\frac{ra_i+1}{a_i+1} - \frac{ra_1+1}{a_1+1} \right) \leq \frac{a_j}{ra_j+1} \sum_{i=2; i \neq j}^r \left(\frac{ra_i+1}{a_i+1} - \frac{ra_1+1}{a_1+1} \right).$$

Therefore, to show the validity of (11), it suffices to show the following stronger inequality:

$$\frac{a_1}{ra_1+1} \left(\sum_{i=2}^r \frac{ra_i+1}{a_i+1} - \sum_{i=2; i \neq j}^r \left(\frac{ra_i+1}{a_i+1} - \frac{ra_1+1}{a_1+1} \right) \right) \leq \frac{a_j}{ra_j+1} \left(\sum_{i=1; i \neq j}^r \frac{ra_i+1}{a_i+1} - \sum_{i=2; i \neq j}^r \left(\frac{ra_i+1}{a_i+1} - \frac{ra_1+1}{a_1+1} \right) \right)$$

which is equivalent to $\frac{a_1}{ra_1+1} \frac{ra_j+1}{a_j+1} + (r-2) \frac{a_1}{a_1+1} \leq (r-1) \frac{a_j}{ra_j+1} \frac{ra_1+1}{a_1+1}$. This is true because

$$\frac{a_1}{ra_1+1} \frac{ra_j+1}{a_j+1} + (r-2) \frac{a_1}{a_1+1} - (r-1) \frac{a_j}{ra_j+1} \frac{ra_1+1}{a_1+1} = -\frac{(r-1)(a_j - a_1)(1 + (r-1)a_1 + a_j)}{(a_1+1)(ra_1+1)(a_j+1)(ra_j+1)} \leq 0.$$

Now we prove (B). In this case, we have $n_i = (1 + \epsilon_i) \frac{ra_i+1}{rk}n$ and $\epsilon = \max_{1 \leq i \leq r} |\epsilon_i|$. Note that $0 \leq \epsilon \leq \frac{1}{2k} \leq \frac{1}{2}$. So we have $\frac{1}{1-\epsilon} \leq 1+2\epsilon \leq (1+\epsilon)^2$ and $(1+\epsilon)^{k+2} \leq \exp(\epsilon(k+2)) \leq 1+(e-1)(k+2)\epsilon$, where we make use of the following facts that $0 \leq \epsilon(k+2) \leq \frac{k+2}{2k} \leq 1$ and $e^x \leq 1+(e-1)x$ for $0 \leq x \leq 1$. Using (10) again, we can derive that

$$\begin{aligned} \frac{\delta(H)}{(1+o(1))\binom{n-1}{k-1}} &\leq \min_{j \in [r]} \binom{k-1}{a_1, a_2, \dots, a_r} \prod_{i=1}^r \left(\frac{ra_i+1}{rk} (1+\epsilon) \right)^{a_i} \cdot \left(1 + \sum_{i=1; i \neq j}^r \frac{(1+\epsilon)a_j(ra_i+1)}{(1-\epsilon)(ra_j+1)(a_i+1)} \right) \\ &\leq \min_{j \in [r]} \binom{k-1}{a_1, a_2, \dots, a_r} (1+\epsilon)^{k-1} \prod_{i=1}^r \left(\frac{ra_i+1}{rk} \right)^{a_i} \cdot (1+\epsilon)^3 \left(1 + \sum_{i=1; i \neq j}^r \frac{a_j(ra_i+1)}{(ra_j+1)(a_i+1)} \right) \\ &= (1+\epsilon)^{k+2} \cdot g_r(\vec{\mathbf{a}}) \leq (1+(e-1)(k+2)\epsilon) \cdot g_r(\vec{\mathbf{a}}) \leq (1+4k\epsilon) \cdot g_r(\vec{\mathbf{a}}), \end{aligned}$$

where the equality holds because of (11) and the definition of $g_r(\vec{\mathbf{a}})$. This finishes the proof. \square

Lemma 15. Let $k \geq 3$ and $r \geq 2$ be integers. For any $\vec{\mathbf{b}} = (b_1, \dots, b_r) \in \mathbb{N}_{k+1}^r$ such that $b_1 \geq 1$, the following hold.

(A) we have $\delta(H(n, \vec{\mathbf{b}})) = (1 + o(1)) \cdot \binom{n-1}{k-1} \cdot \tilde{g}_r(\vec{\mathbf{b}})$, where $o(1) \rightarrow 0$ when $n \rightarrow \infty$ and

$$\tilde{g}_r(\vec{\mathbf{b}}) := \binom{k}{b_1-1, b_2, \dots, b_r} \cdot \prod_{i=2}^r \left(\frac{rb_i-1}{rk} \right)^{b_i} \cdot \left(\frac{rb_1-1}{rk} \right)^{b_1-1} \cdot \left(\frac{r(b_1-1)}{rb_1-1} + \sum_{i=2}^r \frac{rb_i}{rb_i-1} \right).$$

(B) Let $\vec{V} = (V_1, V_2, \dots, V_r)$ be a partition of n vertices such that $\frac{|V_i|}{n} = (1 + \epsilon_i) \frac{rb_i-1}{rk}$. Then

$$\text{by setting } \epsilon = \max_{1 \leq i \leq r} |\epsilon_i| \leq \frac{1}{2k}, \text{ we have } \delta(\tilde{H}(\vec{V}, \vec{\mathbf{b}})) \leq (1 + 4k\epsilon + o(1)) \cdot \binom{n-1}{k-1} \cdot \tilde{g}_r(\vec{\mathbf{b}}).$$

Proof. In this proof, we denote H as $\tilde{H}(n, \vec{\mathbf{b}})$, $\tilde{H}(\vec{V}, \vec{\mathbf{b}})$, and we define $V_1 \cup V_2 \cup \dots \cup V_r$ as the default r -partition of H . By definition, for any $e \in E(H)$, there exists some $i \in [r]$ such that $|e \cap V_i| = b_i - 1$ and $|e \cap V_j| = b_j$ for all $j \in [r] \setminus \{i\}$. Let $E_i(H)$ be the set of all edges e in H with $|e \cap V_i| = b_i - 1$. Let $n_i = |V_i|$ for $i \in [r]$. Suppose that each $n_i \rightarrow \infty$ as $n \rightarrow \infty$. Then in both cases, we have⁴.

$$\begin{aligned} \delta(H) &= \min_{v \in V(H)} \sum_{\ell=1}^r \sum_{e \in E_\ell(H)} 1_{v \in e} \\ &= \min_{j \in [r]} \sum_{\ell=1; \ell \neq j}^r \left(\prod_{i=1; i \neq j, \ell}^r \binom{n_i}{b_i} \right) \binom{n_\ell}{b_\ell-1} \binom{n_j-1}{b_j-1} + \left(\prod_{i=1; i \neq j}^r \binom{n_i}{b_i} \right) \binom{n_j-1}{b_j-2} \\ &= \left(\prod_{i=1; i \neq j}^r \binom{n_i}{b_i} \right) \binom{n_j-1}{b_j} \left(\frac{b_j(b_j-1)}{(n_j-b_j+1)(n_j-b_j)} + \frac{b_j}{n_j-b_j} \sum_{\ell=1; \ell \neq j}^r \frac{b_\ell}{n_\ell-b_\ell+1} \right), \end{aligned}$$

thus implying that (here $o(1)$ denotes that $o(1) \rightarrow 0$ as $n \rightarrow \infty$)

$$\frac{\delta(H)}{(1+o(1))\binom{n-1}{k-1}} \tag{12}$$

$$= \frac{1}{k(k+1)} \cdot \min_{j \in [r]} \binom{k+1}{b_1, b_2, \dots, b_r} \left(\prod_{i=1}^r \left(\frac{n_i}{n} \right)^{b_i} \right) \left(b_j(b_j-1) \cdot \left(\frac{n}{n_j} \right)^2 + \sum_{\ell=1, \ell \neq j}^r b_j b_\ell \cdot \left(\frac{n^2}{n_j n_\ell} \right) \right). \tag{13}$$

We first prove (A). In this case, n is divisible by rk and $n_i = \frac{rb_i-1}{rk}n$. Hence, we derive from (12)

$$\begin{aligned} \frac{\delta(H)}{(1+o(1))\binom{n-1}{k-1}} &= \frac{1}{k(k+1)} \cdot \min_{j \in [r]} \binom{k+1}{b_1, b_2, \dots, b_r} \prod_{i=1}^r \left(\frac{rb_i-1}{rk} \right)^{b_i} \\ &\quad \cdot \left(b_j(b_j-1) \cdot \left(\frac{rk}{rb_j-1} \right)^2 + \sum_{\ell=1; \ell \neq j}^r b_j b_\ell \cdot \left(\frac{rk}{rb_j-1} \right) \left(\frac{rk}{rb_\ell-1} \right) \right) \\ &= \frac{r^2k}{k+1} \cdot \min_{j \in [r]} \binom{k+1}{b_1, b_2, \dots, b_r} \prod_{i=1}^r \left(\frac{rb_i-1}{rk} \right)^{b_i} \left(\frac{b_j}{rb_j-1} \left(\frac{b_j-1}{rb_j-1} + \sum_{\ell=1; \ell \neq j}^r \frac{b_\ell}{rb_\ell-1} \right) \right). \end{aligned}$$

Recall that $1 \leq b_1 \leq b_2 \leq \dots \leq b_r$. To prove (A), it remains to show that for all $2 \leq j \leq r$,

$$\frac{b_1}{rb_1-1} \left(\frac{b_1-1}{rb_1-1} + \sum_{\ell=2}^r \frac{b_\ell}{rb_\ell-1} \right) \leq \frac{b_j}{rb_j-1} \left(\frac{b_j-1}{rb_j-1} + \sum_{\ell=1; \ell \neq j}^r \frac{b_\ell}{rb_\ell-1} \right) \tag{14}$$

⁴We admit that $\binom{n}{m} = 0$ for $m < 0$ or $m > n$.

Observing that for $1 \leq a \leq b$, it holds that $\frac{a}{ra-1} \geq \frac{b}{rb-1}$, we can deduce that

$$\begin{aligned} & \frac{b_1}{rb_1-1} \left(\frac{b_1-1}{rb_1-1} + \sum_{\ell=2}^r \frac{b_\ell}{rb_\ell-1} \right) - \frac{b_j}{rb_j-1} \left(\frac{b_j-1}{rb_j-1} + \sum_{\ell=1; \ell \neq j}^r \frac{b_\ell}{rb_\ell-1} \right) \\ & \leq \frac{b_1(b_1-1)}{(rb_1-1)^2} - \frac{b_j(b_j-1)}{(rb_1-1)(rb_j-1)} + (r-2) \left(\frac{b_1}{rb_1-1} - \frac{b_j}{rb_j-1} \right) \frac{b_1}{rb_1-1} \\ & = \frac{(b_1-b_j)((r-1)b_1+b_j-1)}{(rb_1-1)^2(rb_j-1)^2} \leq 0. \end{aligned}$$

Hence

$$\begin{aligned} \frac{\delta(H)}{\binom{n-1}{k-1}} &= (1+o(1)) \frac{r^2 k}{k+1} \binom{k+1}{b_1, b_2, \dots, b_r} \prod_{i=1}^r \left(\frac{rb_i-1}{rk} \right)^{b_i} \left(\frac{b_1}{rb_1-1} \left(\frac{b_1-1}{rb_1-1} + \sum_{i=2}^r \frac{b_i}{rb_i-1} \right) \right) \\ &= (1+o(1)) \cdot \binom{k}{b_1-1, b_2, \dots, b_r} \prod_{i=2}^r \left(\frac{rb_i-1}{b_i} \right)^{b_i} \cdot \left(\frac{rb_1-1}{rk} \right)^{b_1-1} \cdot \left(\frac{r(b_1-1)}{rb_1-1} + \sum_{i=2}^r \frac{rb_i}{rb_i-1} \right) \end{aligned}$$

finishing the proof of item (A).

Now we prove (B). In this case, we have $n_i = (1+\epsilon_i) \frac{rb_i-1}{rk} n$ and $\epsilon = \max_{1 \leq i \leq r} |\epsilon_i|$. Note that $0 \leq \epsilon \leq \frac{1}{2k} \leq \frac{1}{2}$. So we have $\frac{1}{1-\epsilon} \leq 1+2\epsilon \leq (1+\epsilon)^2$ and $(1+\epsilon)^{\frac{rk}{k+2}} \leq \exp(\epsilon(k+2)) \leq 1+(e-1)(k+2)\epsilon$, where we make use of the following facts that $0 \leq \epsilon(k+2) \leq \frac{k+2}{2k} \leq 1$ and $e^x \leq 1+(e-1)x$ for $0 \leq x \leq 1$. Using (12) again, let δ_{ij} be the Kronecker notation such that $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$, we can derive that

$$\begin{aligned} & \frac{\delta(H)}{(1+o(1)) \binom{n-1}{k-1}} \\ &= \frac{1}{k(k+1)} \cdot \min_{j \in [r]} \binom{k+1}{b_1, b_2, \dots, b_r} \left(\prod_{i=1}^r \binom{n_i}{n}^{b_i} \right) \left(b_j(b_j-1) \cdot \binom{n}{n_j}^2 + \sum_{\ell=1, \ell \neq j}^r b_j b_\ell \cdot \binom{n^2}{n_j n_\ell} \right) \\ &= \frac{1}{k} \cdot \min_{j \in [r]} \binom{k}{b_1, \dots, b_j-1, \dots, b_r} \left(\prod_{i=1}^r \binom{n_i}{n}^{b_i - \delta_{ij}} \right) \left((b_j-1) \cdot \binom{n}{n_j} + \sum_{\ell=1; \ell \neq j}^r b_\ell \cdot \binom{n}{n_\ell} \right) \\ &\leq \min_{j \in [r]} \binom{k}{b_1, \dots, b_j-1, \dots, b_r} \prod_{i=1}^r \left(\frac{rb_i-1}{rk} (1+\epsilon) \right)^{b_i - \delta_{ij}} \\ &\quad \cdot \left((b_j-1) \cdot \frac{1}{k} \cdot \left(\frac{rk}{(1-\epsilon)(rb_j-1)} \right) + \sum_{\ell=1; \ell \neq j}^r b_\ell \cdot \frac{1}{k} \cdot \left(\frac{rk}{(1-\epsilon)(rb_\ell-1)} \right) \right) \\ &= \min_{j \in [r]} \binom{k}{b_1, \dots, b_j-1, \dots, b_r} \prod_{i=1}^r \left(\frac{rb_i-1}{rk} (1+\epsilon) \right)^{b_i - \delta_{ij}} \cdot \frac{1}{1-\epsilon} \cdot \left(\frac{r(b_j-1)}{rb_j-1} + \sum_{\ell=1; \ell \neq j}^r \frac{rb_\ell}{rb_\ell-1} \right) \\ &\leq \frac{(1+\epsilon)^k}{1-\epsilon} \cdot \tilde{g}_r(\vec{\mathbf{b}}) \leq (1+\epsilon)^{k+2} \cdot \tilde{g}_r(\vec{\mathbf{b}}) \leq (1+(e-1)(k+2)\epsilon) \cdot \tilde{g}_r(\vec{\mathbf{b}}) \leq (1+4k\epsilon) \cdot \tilde{g}_r(\vec{\mathbf{b}}), \end{aligned}$$

where the equality holds because of (14) and the definition of $\tilde{g}_r(\vec{\mathbf{b}})$. This finishes the proof. \square

In the following lemma, we will see that for $r \geq 3$ and $k \geq 3$, $\lim_{n \rightarrow \infty} \frac{\delta(H(n, \vec{\mathbf{b}}))}{\binom{n-1}{k-1}} < \max\{f(k), g_r(k)\}$. Furthermore, it suggests that $\tilde{H}^*(n, \vec{\mathbf{b}})$ will not be the extreme graph with low perfect matching discrepancy and high degree. The proof can be found in Appendix A.

Lemma 16. *For any $r \geq 3$, $k \geq 3$ and $\vec{\mathbf{b}} \in \mathbb{N}_{k+1}^r$ satisfying $b_1 \geq 1$, then the following hold:*

(A) If $b_1 = 1$, then for every positive integer n is divisible by rk , $\tilde{H}(n, \vec{\mathbf{b}})$ is a subgraph of $H(n, (0, \dots, 0, k-1))$, implying that $\tilde{g}_r(\vec{\mathbf{b}}) \leq g_r(k) \leq \max\{f(k), g_r(k)\}$;

(B) If $b_1 \geq 2$, then

$$\tilde{g}_r(\vec{\mathbf{b}}) < f(k) \leq \max\{f(k), g_r(k)\},$$

4 Proof of Theorem 4

For fixed integers $k \geq 3$ and $r \geq 2$, we take $\gamma = \gamma(k, r) > 0$ to be sufficiently small.⁵ Given any $\eta > 0$,⁶ let $n_0 = n_0(r, k, \eta)$ be sufficiently large. Let H be a k -uniform hypergraph on $n \geq n_0$ vertices with an r -edge-coloring $c : E(H) \rightarrow \{C_1, C_2, \dots, C_r\}$ such that $n \equiv 0 \pmod{k}$ and

$$\delta(H) > (\max\{f(k), g_r(k)\} + \eta) \cdot \binom{n-1}{k-1}.$$

Suppose for a contradiction that every perfect matching of H has less than $\frac{n}{rk}(1 + \gamma \cdot \eta)$ edges with the color C_i for any $i \in [r]$. For a subgraph F of H , let $C_i(F)$ denote the number of C_i -edges contained in F . As a consequence, we can derive that for every perfect matching \mathcal{M} of H and every $i \in [r]$,

$$\frac{n}{rk} \left(1 - (r-1)\gamma \cdot \eta\right) < C_i(\mathcal{M}) < \frac{n}{rk} \left(1 + \gamma \cdot \eta\right). \quad (15)$$

First we consider when H contains $t = \eta n / (20 \cdot rk^2(k+1))$ (vertex-)disjoint good gadgets (of any kind), denoted by G_1, \dots, G_t . For $j \in [t]$, let \mathcal{M}_j and \mathcal{N}_j be two perfect matchings of G_j with different color profiles. Let $H' = H - \cup_{j=1}^t V(G_j)$. Since each G_i has at most $k^2 + k$ vertices, we have

$$\delta(H') \geq \delta(H) - t(k^2 + k) \binom{n-2}{k-2} > (\max\{f(k), g_r(k)\} + \eta/2) \cdot \binom{n-1}{k-1}. \quad (16)$$

Note that $|V(H')|$ is divisible by k . By the definition of $f(k)$, H' has a perfect matching say \mathcal{L} . Since every \mathcal{M}_j and \mathcal{N}_j have different color profiles, by average there exists some $i \in [r]$ (let us say $i = 1$) such that at least t/r many indices $j \in [t]$ satisfy $C_1(\mathcal{M}_j) \neq C_1(\mathcal{N}_j)$. Renaming if needed we may assume that for all $j \in [t]$, we have $C_1(\mathcal{M}_j) \geq C_1(\mathcal{N}_j)$, and thus we obtain that

$$\sum_{j=1}^t C_1(\mathcal{M}_j) \geq \sum_{j=1}^t C_1(\mathcal{N}_j) + \frac{t}{r}.$$

Let $\mathcal{M} = \mathcal{L} \cup \mathcal{M}_1 \cup \dots \cup \mathcal{M}_t$ and $\mathcal{N} = \mathcal{L} \cup \mathcal{N}_1 \cup \dots \cup \mathcal{N}_t$ be two perfect matchings of H . Then

$$C_1(\mathcal{M}) - C_1(\mathcal{N}) = \sum_{j=1}^t (C_1(\mathcal{M}_j) - C_1(\mathcal{N}_j)) \geq \frac{t}{r} = \frac{\eta n}{20 \cdot r^2 k^2 (k+1)}.$$

On the other hand, using (15) we have $C_1(\mathcal{M}) - C_1(\mathcal{N}) \leq \frac{\gamma}{k} \eta n$, which is a contradiction as $\gamma = \gamma(k, r)$ is chosen to be sufficiently small. This proves the first case.

Now we may assume that H contains less than $\eta n / (20 \cdot rk^2(k+1))$ disjoint good gadgets. We can greedily choose t disjoint good gadgets, denoted by G_1, \dots, G_t , such that $H' := H - \cup_{i=1}^t V(G_i)$ contains no good gadgets of any kind, where $t < \eta n / (20 \cdot rk^2(k+1))$. Let $m = |\cup_{i=1}^t V(G_i)|$. So

$$m \leq t(k^2 + k) < \frac{\eta n}{20 \cdot rk}.$$

⁵Our proof shows that γ can be chosen as $\gamma = \Theta(1/r^2 k^2)$.

⁶We may assume that $0 < \eta < \frac{1}{2}$, because otherwise the inequality regarding the minimum vertex degree of H below would not hold.

Also note that $|V(H')|$ is divisible by k and moreover, the same inequality (16) holds for H' . Hence, H' contains at least one perfect matching. Let \mathcal{L} be any perfect matching of H' . If \mathcal{L} contains at least $\frac{|V(H')|}{rk} + \frac{m}{rk} + \frac{\gamma\eta m}{rk}$ edges of the same color, then, when combined with any perfect matching of G_i for all $i \in [t]$, it forms a perfect matching of H which contradicts (15). Hence we may assume that

$$C_i(\mathcal{L}) < \frac{|V(H')|}{rk} + \left(\frac{m}{rk} + \frac{\gamma\eta m}{rk}\right) \leq \frac{|V(H')|}{rk} + \frac{\eta \cdot |V(H')|}{15 \cdot r^2 k^2} \text{ for any } i \in [r],$$

where the last inequality holds because $m < \frac{\eta m}{20 \cdot rk}$ and γ is sufficiently small. It further implies that

$$\frac{|V(H')|}{rk} - \frac{(r-1)\eta \cdot |V(H')|}{15 \cdot r^2 k^2} < C_i(\mathcal{L}) < \frac{|V(H')|}{rk} + \frac{\eta \cdot |V(H')|}{15 \cdot r^2 k^2} \text{ for each } i \in [r]. \quad (17)$$

In particular, it shows that H' has at least one C_i -edge for each $i \in [r]$. Using (16), it is also easy to derive that $\delta(H') > \frac{1}{2} \binom{n-1}{k-1} + \frac{k^2+1}{2} \binom{n-2}{k-2}$. Now we see that H' satisfies all conditions of Lemma 9. By Lemma 9, we first consider that there exists an r -partition $\vec{V}' = (V'_1, \dots, V'_r)$ of $V(H')$ and a vector $\vec{\mathbf{a}} \in \mathbb{N}_{k-1}^r$ such that H' is an r -edge-colored subgraph of $H^*(\vec{V}', \vec{\mathbf{a}})$. Any perfect matching \mathcal{L} of H' is of course a perfect matching of $H^*(\vec{V}', \vec{\mathbf{a}})$. Now recall the observation (4), which asserts that

$$|V'_i| = a_i \frac{|V(H')|}{k} + C_i(\mathcal{L}) \text{ for each } i \in [r].$$

Let $|V'_i|/|V(H')| = (1 + \epsilon_i) \frac{ra_i+1}{rk}$. Then the above equality, along with (17), implies that

$$\epsilon := \max_{1 \leq i \leq r} |\epsilon_i| \leq \max_{1 \leq i \leq r} \frac{(r-1)\eta}{15 \cdot rk \cdot (ra_i+1)} \leq \frac{\eta}{15k} \leq \frac{1}{2k}, \quad (18)$$

where we use $\eta < \frac{1}{2}$. By item (B) of Lemma 14, since n is sufficiently large, we can derive that

$$\begin{aligned} \delta(H') &\leq \delta(H(\vec{V}', \vec{\mathbf{a}})) \leq (1 + 4k\epsilon + \frac{\eta}{15}) \cdot \binom{|V(H')| - 1}{k-1} \cdot g_r(\vec{\mathbf{a}}) \\ &\leq \left(1 + \frac{\eta}{3}\right) \cdot \binom{n-1}{k-1} \cdot g_r(k) \leq \left(g_r(k) + \frac{\eta}{3}\right) \cdot \binom{n-1}{k-1}, \end{aligned}$$

where the second last inequality holds because of (18) and the definition of $g_r(k)$, and the last inequality follows from the fact that $g_r(k) \leq 1$. On the other hand, by (16) we have $\delta(H') \geq (g_r(k) + \frac{\eta}{2}) \cdot \binom{n-1}{k-1}$, which is a contradiction.

Now we consider the case of H' is a r -edge-colored subgraph of $\tilde{H}^*(\vec{V}', \vec{\mathbf{b}})$ for some partition \vec{V}' of V' and $\vec{\mathbf{b}} \in \mathbb{N}_{k+1}^r$ such that $b_1 \geq 1$. If $r = 2$, then $\tilde{H}((V_1, V_2), (b_1, b_2)) = H((V_1, V_2), (b_1 - 1, b_2 - 1))$. Hence it leads to a contradiction by previous discussion. So we may assume that $r \geq 3$. Then by item (B) of Lemma 15 and Lemma 16, following the same discussion, we have

$$\delta(H') \leq \left(\tilde{g}_r(b_1, \dots, b_r) + \frac{\eta}{3}\right) \cdot \binom{n-1}{k-1} \leq \left(\max\{f(k), g_r(k)\} + \frac{\eta}{3}\right) \cdot \binom{n-1}{k-1},$$

which contradicts with $\delta(H') \geq (\max\{f(k), g_r(k)\} + \frac{\eta}{2}) \cdot \binom{n-1}{k-1}$. This completes the proof of Theorem 4. \square

5 Proof of Theorem 5

The goal of this section is to establish the proof of Theorem 5. Throughout this section, we denote the function $f_0(k) = 1 - (1 - \frac{1}{k})^{k-1}$. Recall that we have

$$f(k) = \limsup_{n \rightarrow \infty} \frac{m(k, n)}{\binom{n-1}{k-1}} \geq f_0(k) \quad \text{and} \quad g_r(k) = \max_{\vec{\mathbf{a}} \in \mathbb{N}_{k-1}^r} g_r(\vec{\mathbf{a}}), \quad (19)$$

where the first inequality becomes an equality when $k \in \{2, 3, 4, 5\}$, and the second expression is given by Lemma 14.

In the following lemma, we present some sufficient conditions under which $f(k) > g_r(k)$ holds. The proof can be found in Appendix B.

Lemma 17. *Let $k \geq 3$ and $r \geq 2$ be integers. Then $f(k) > g_r(k)$ if one of the following holds:*

- $r = 2$ and $k \geq 20$;
- $r \geq 3$ and $k \geq 10$; and
- $r \geq 6$ and $k \geq 3$.

Using this lemma, we are ready to present the proof of Theorem 5.

Proof of Theorem 5. By Proposition 3 and Theorem 4, we can derive that $h_r(k) = \max\{f(k), g_r(k)\}$ for all $k \geq 3$ and $r \geq 2$. By Lemma 17, we see that $f(k) > g_r(k)$ holds whenever one of the following holds: (I) $r = 2$ and $k \geq 20$; (II) $r \geq 3$ and $k \geq 10$, and (III) $r \geq 6$ and $k \geq 3$.

To complete the proof, it suffices to examine the following uncovered cases: (A) $r = 2$ and $3 \leq k \leq 19$ and (B) $3 \leq r \leq 5$ and $3 \leq k \leq 9$. These cases are finite in number, allowing us to determine the precise values of the corresponding constants. Let $f_0(k) = 1 - (1 - \frac{1}{k})^{k-1}$. Recall that $f(k) \geq f_0(k)$, with equality if $k \in \{3, 4, 5\}$. In the following two tables, we compare with the approximate values of $g_r(k)$ and $f_0(k)$ for the cases (A) and (B), respectively. Numerical details for Table 1 and Table 2 can be found in Appendix C.

k	3	4	5	6	7	8	9	10	11
$g_2(k)$	0.75 ⁷	0.6836 ⁸	0.6561 ⁹	0.6472	0.6410	0.6365	0.6330	0.6302	0.6280
$f_0(k)$	0.5556	0.5781	0.5904	0.5981	0.6034	0.6073	0.6103	0.6126	0.6145

k	12	13	14	15	16	17	18	19
$g_2(k)$	0.6262	0.6246	0.6233	0.6221	0.6211	0.6202	0.6195	0.6188
$f_0(k)$	0.6160	0.6173	0.6184	0.6194	0.6202	0.6209	0.6216	0.6221

Table 1: The values of $g_2(k)$ and $f_0(k)$ for $3 \leq k \leq 19$

k	3	4	5	6	7	8	9
$g_3(k)$	0.6049 ¹⁰	0.5787 ¹¹	0.5642	0.5549	0.5485	0.5439	0.5403
$g_4(k)$	0.5625 ¹²	0.5363	0.5220	0.5129	0.5066	0.5020	0.4985
$g_5(k)$	0.5378	0.512	0.4979	0.4889	0.4828	0.4783	0.4749
$f_0(k)$	0.5556	0.5781	0.5904	0.5981	0.6034	0.6073	0.6103

Table 2: The values of $g_r(k)$ and $f_0(k)$ for $3 \leq r \leq 5$ and $3 \leq k \leq 9$

In Table 1, the values of $g_2(k)$ for $3 \leq k \leq 5$ (highlighted in green) indicate that $g_2(k) > f_0(k) = f(k)$, the values of $g_2(k)$ for $17 \leq k \leq 19$ (highlighted in blue) indicate that $f(k) \geq f_0(k) > g_2(k)$, and the other values present situations where a comparison between $g_2(k)$ and $f(k)$ is infeasible. In Table 2, the values of $g_r(k)$ for $(r, k) \in \{(3, 3), (3, 4), (4, 3)\}$ (highlighted in red) indicate that $g_r(k) >$

⁷The value of $g_2(3)$ is maximized by the vector $\vec{a} = (1, 1)$.

⁸The value of $g_2(4)$ is maximized by the vector $\vec{a} = (1, 2)$.

⁹The value of $g_2(5)$ is maximized by the vector $\vec{a} = (0, 4)$.

¹⁰The value of $g_3(3)$ is maximized by the vector $\vec{a} = (0, 0, 2)$.

¹¹The value of $g_3(4)$ is maximized by the vector $\vec{a} = (0, 0, 3)$.

¹²The value of $g_4(3)$ is maximized by the vector $\vec{a} = (0, 0, 0, 2)$.

$f_0(k) = f(k)$, while all other values of $g_r(k)$ demonstrate that $f(k) \geq f_0(k) > g_r(k)$. In particular, each footnote indicates the unique maximum vector \vec{a} for the corresponding $g_r(k) = \max_{\vec{a} \in \mathbb{N}_{k-1}^r} g_r(\vec{a})$.

Taken together with all the information above, this leads to the assertions in Theorem 5 for $f(k) > g_r(k)$ and $g_r(k) > f(k)$, respectively. \square

6 Concluding remarks

In this paper, we establish the minimum vertex degree threshold of perfect matchings with high r -color discrepancy in k -uniform hypergraphs, by showing that $h_r(k) = \max\{f(k), g_r(k)\}$ for all $k \geq 3$ and $r \geq 2$. Our proofs have the potential to unveil further insights, and we would like to offer some additional remarks on this matter.

- It should be noted that if the matching existence conjecture (i.e., (1) is conjectured to be an equality) holds true for $6 \leq k \leq 16$, our proofs would lead to the conclusion that $h_2(k) = g_2(k)$ for $6 \leq k \leq 16$. In these 11 instances, in addition to the 6 cases outlined in Theorem 5 under the condition $g_r(k) > f(k)$, the conclusion $h_r(k) = g_r(k)$ would exclusively apply, while all other cases would follow $h_r(k) = f(k)$.
- In all cases where $g_r(k) > f(k)$ occurs, our proofs can be easily modified into a stability result, with the corresponding $H^*(n, \vec{a})$ as the r -edge-colored extremal hypergraph, where \vec{a} denotes the unique maximum vector for the optimization $g_r(k) = \max_{\vec{a} \in \mathbb{N}_{k-1}^r} g_r(\vec{a})$.
- The key technical Lemma 9 (as well as other lemmas in Subsection 3.1 under the good-gadget-free condition) holds without requiring the existence of perfect matchings or the condition $n \equiv 0 \pmod{k}$. So our proofs can also help illuminate the establishment of the corresponding threshold for a *near-perfect matching* (i.e., a matching of size $\frac{n}{k} - O_k(1)$ in n -vertex k -graphs) with high discrepancy. For a more in-depth discussion, we recommend readers refer to the pertinent paragraph in the concluding section of Gishboliner-Glock-Sgueglia [12].
- We have seen that in k -graphs for $k \geq 3$, the ℓ -degree discrepancy threshold $h_r^\ell(k)$ equals the ℓ -degree existence threshold $f^\ell(k)$ of perfect matchings for all but a finite number of cases. Recall the well-known matching conjecture that $f^\ell(k) = \max\left\{\frac{1}{2}, 1 - \left(1 - \frac{1}{k}\right)^{k-\ell}\right\}$. Constructions that result in the constant $\frac{1}{2}$ are commonly referred to as *parity obstacles*. Also recall the definition of $m_\ell(k, n)$. We wonder if in the case that $r = 2$, $\ell = k - 1$ and $k \geq 5$ (note that $h_2^{k-1}(k) = f^{k-1}(k) = \frac{1}{2}$ is determined by the parity obstacle), there exist positive constants $c = c(k, \ell)$ and $C = C(k, \ell)$ such that for sufficiently large n with $n \equiv 0 \pmod{k}$, any 2-edge-colored n -vertex k -graph H with $\delta_{k-1}(H) \geq m_{k-1}(k, n) + C$ contains a perfect matching with at least $\frac{n}{2k} + c \cdot n$ edges with the same color. In essence, we suspect that in these cases, once a perfect matching emerges, a perfect matching with high discrepancy appears almost immediately.

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Appendices

A Proof of Lemma 16

In this section, we give a detailed proof of Lemma 16.

We will first establish some basic inequalities in the following two propositions. These propositions will be frequently used in the numerical proofs in this paper. Throughout, we use $\exp(x)$ to denote the exponential function e^x .

Proposition 18. *Let k be a positive integer. Then the following hold:*

$$(a) \text{ For } k \geq 3, \left(1 - \frac{1}{k}\right)^{k-1} \leq \left(1 - \frac{1}{3}\right)^2 = \frac{4}{9} \leq 1.21e^{-1};$$

(b) For $k \geq 10$, $(1 - \frac{1}{k})^{k-1} \leq (1 - \frac{1}{10})^9 \leq 1.06e^{-1}$;

(c) For $k \geq 20$, $(1 - \frac{1}{k})^{k-1} \leq (1 - \frac{1}{20})^{19} \leq 0.378$, $(1 - \frac{3}{2k})^{k-2} \leq (1 - \frac{3}{40})^{18} \leq 0.246$ and $(1 - \frac{1}{2k})^{k-1} \leq (1 - \frac{1}{40})^{19} \leq 0.619$;

(d) For $k \geq 3$, if $r \geq 3$ and $1 \leq m \leq k - 2$ are integers, then

$$\sqrt{\frac{k-1}{k-m-1}} \left(\frac{1}{2\pi}\right)^{\frac{m}{2}} \leq \sqrt{2m} \left(\frac{1}{2\pi}\right)^{\frac{m}{2}} \leq \exp\left(-\frac{m}{3}\right) \leq \exp\left(-\frac{m}{r}\right).$$

Proof. We first show that $h_1(k) = (1 - \frac{1}{k})^{k-1}$ is decreasing over \mathbb{N}^+ . Indeed, it follows as

$$\frac{h_1(k)}{h_1(k+1)} = \frac{(k^2 - 1)^{k-1}(k+1)}{k^{2k-1}} = \frac{(1 - \frac{1}{k^2-1})^{k-1}}{1 - \frac{1}{k+1}} \geq \frac{1 - \frac{k-1}{k^2-1}}{1 - \frac{1}{k+1}} = 1.$$

Hence by simple calculation, we can directly derive (a), (b) and the first assertion of (c). Similarly, we can show that $h_2(k) = (1 - \frac{3}{2k})^{k-2}$ and $h_3(k) = (1 - \frac{1}{2k})^{k-1}$ are decreasing for $k \geq 2$:

$$\frac{h_2(k)}{h_2(k+1)} = \frac{2k+2}{2k-1} \left(\frac{4k^2 - 2k - 6}{4k^2 - 2k}\right)^{k-2} = \frac{(1 - \frac{3}{k(2k-1)})^{k-2}}{1 - \frac{3}{2k+2}} \geq \frac{1 - \frac{3(k-2)}{k(2k-1)}}{1 - \frac{3}{2k+2}} \geq 1$$

$$\frac{h_3(k)}{h_3(k+1)} = \frac{2k+2}{2k+1} \left(\frac{4k^2 + 2k - 2}{4k^2 + 2k}\right)^{k-1} = \frac{(1 - \frac{1}{k(2k+1)})^{k-1}}{1 - \frac{1}{2k+2}} \geq \frac{1 - \frac{k-1}{k(2k+1)}}{1 - \frac{1}{2k+2}} \geq 1.$$

Hence for $k \geq 20$, we have $(1 - \frac{3}{2k})^{k-2} \leq (1 - \frac{3}{40})^{18} \leq 0.246$ and $(1 - \frac{1}{2k})^{k-1} \leq (1 - \frac{1}{40})^{19} \leq 0.619$. This completes the proof for (c). For (d), since $m(k-m-1) \geq k-2$, we have $\frac{k-1}{k-m-1} \leq \frac{m(k-1)}{k-2} \leq 2m$, which implies the first inequality of (d). Let $\gamma(m) = \sqrt{2m}(\frac{e^{1/3}}{\sqrt{2\pi}})^m$. Then it suffices to show $\gamma(m) \leq 1$. To see it, we have $\gamma(1) = 0.7874 < 1$ by calculate directly. Observe that for all $m \in \mathbb{N}^+$

$$\frac{\gamma(m+1)}{\gamma(m)} = \sqrt{\frac{m+1}{m}} \frac{e^{1/3}}{\sqrt{2\pi}} \leq \frac{\sqrt{2}e^{1/3}}{\sqrt{2\pi}} = 0.7874 < 1,$$

so $\gamma(m) \leq \gamma(1) < 1$ holds for all positive integer m , completing the proof. \square

We will also use Stirling's Formula to provide the following estimations on the factorials.

Proposition 19. For all $k \in \mathbb{N}^+$, $k! > \sqrt{2\pi}(\frac{k}{e})^k \sqrt{k}$. Moreover, we have

$$\frac{k!}{\sqrt{2\pi}(\frac{k}{e})^k \sqrt{k}} \leq 1.01 \text{ for } k \geq 9, \text{ and } \frac{k!}{\sqrt{2\pi}(\frac{k}{e})^k \sqrt{k}} \leq 1.05 \text{ for } k \geq 2.$$

Proof. Let $h(k) = \frac{k!}{(\frac{k}{e})^k \sqrt{k}}$. Then we have $\frac{h(k+1)}{h(k)} = \frac{e(k+1)}{(k+1)(1+\frac{1}{k})^{k+\frac{1}{2}}} = \frac{e}{(1+\frac{1}{k})^{k+\frac{1}{2}}} < 1$, thereby $h(k)$ is decreasing. On the other hand, using Stirling's Formula, we see $\lim_{k \rightarrow \infty} h(k) = \sqrt{2\pi}$. Hence $h(k) > \sqrt{2\pi}$ for all $k \in \mathbb{N}^+$, which implies that $k! > \sqrt{2\pi}(\frac{k}{e})^k \sqrt{k}$. For $k \geq 9$, we have $h(k) \leq h(9) \leq 1.01\sqrt{2\pi}$, which implies that $\frac{k!}{\sqrt{2\pi}(\frac{k}{e})^k \sqrt{k}} \leq 1.01$ for $k \geq 9$. Similarly, we have $h(k) \leq h(2) \leq 1.05\sqrt{2\pi}$ for $k \geq 2$, indicating that $\frac{k!}{\sqrt{2\pi}(\frac{k}{e})^k \sqrt{k}} \leq 1.05$ for $k \geq 2$. \square

Now we are ready to prove some numerical conclusions using these tools.

proof of Lemma 16. Recall that $r \geq 3, k \geq 3, 1 \leq b_1 \leq \dots \leq b_r$ and $k + 1 = b_1 + \dots + b_r$.

Firstly, we prove item (A). In this case $b_1 = 1$. Suppose that (V_1, \dots, V_r) is the vertex partition of $H = \tilde{H}(n, (1, b_2, \dots, b_r))$ such that $|V_i| = \frac{rb_i - 1}{rk}n$ for $i \in [r]$. Then by definition, we have

$$E(\tilde{H}(n, (1, b_2, \dots, b_r))) \subseteq \binom{V_1}{1} \times \binom{V_2 \cup \dots \cup V_r}{k-1} \cup \binom{V_2 \cup \dots \cup V_r}{k}.$$

Let V'_1, \dots, V'_{r-1} be a partition of V_1 such that $|V'_i| = \frac{n}{rk}$ for $1 \leq i \leq r-1$ and let $V'_r = V_2 \cup \dots \cup V_r$. Then $|V'_r| = \frac{r(k-1)+1}{rk}n$ and $H((V'_1, \dots, V'_r), (0, \dots, 0, k-1)) = \binom{V_1}{1} \times \binom{V_2 \cup \dots \cup V_r}{k-1} \cup \binom{V_2 \cup \dots \cup V_r}{k}$. Hence

$$\tilde{H}(n, (1, b_2, \dots, b_r)) \subseteq H((V'_1, \dots, V'_r), (0, \dots, 0, k-1)) \cong H(n, (0, \dots, 0, k-1)),$$

finishing the proof of (A).

Now we prove (B). In this case $b_1 \geq 2$ and $k = b_1 + \dots + b_r - 1 \geq 2r - 1 \geq 5$. By the proof of Proposition 18, $h_1(k) = (1 - \frac{1}{k})^{k-1}$ is decreasing over \mathbb{N}^+ , hence

$$f(k) \geq f_0(k) = 1 - (1 - \frac{1}{k})^{k-1} \geq 1 - (1 - \frac{1}{5})^{5-1} = 0.5904.$$

By the observation that $\frac{b_i}{rb_i - 1} \leq \frac{b_1}{rb_1 - 1}$ for all $i \geq 2$, we have

$$\frac{b_1 - 1}{rb_1 - 1} + \sum_{i=2}^r \frac{b_i}{rb_i - 1} \leq \frac{b_1 - 1}{rb_1 - 1} + (r-1) \frac{b_1}{rb_1 - 1} = 1. \quad (20)$$

Using inequality (20) and Proposition 19, we have

$$\begin{aligned} \tilde{g}_r(\vec{\mathbf{b}}) &= \binom{k}{b_1 - 1, b_2, \dots, b_r} \cdot \prod_{i=2}^r \left(\frac{rb_i - 1}{rk} \right)^{b_i} \cdot \left(\frac{rb_i - 1}{rk} \right)^{b_1 - 1} \cdot \left(\frac{r(b_1 - 1)}{rb_1 - 1} + \sum_{i=2}^r \frac{rb_i}{rb_i - 1} \right) \\ &\leq r \cdot \binom{k}{b_1 - 1, b_2, \dots, b_r} \cdot \prod_{i=2}^r \left(\frac{rb_i - 1}{rk} \right)^{b_i} \cdot \left(\frac{rb_i - 1}{rk} \right)^{b_1 - 1} \\ &\leq 1.05r \cdot \frac{\sqrt{2\pi} \cdot k^{k + \frac{1}{2}}}{\sqrt{2\pi} \cdot (b_1 - 1)^{b_1 - 1 + \frac{1}{2}} \prod_{i=2}^r \left(\sqrt{2\pi} \cdot b_i^{b_i + \frac{1}{2}} \right)} \cdot \prod_{i=2}^r \left[\left(\frac{b_i}{k} \right)^{b_i} \cdot \left(1 - \frac{1}{rb_i} \right)^{b_i} \right] \\ &\quad \cdot \left(\frac{b_1 - 1}{k} \right)^{b_1 - 1} \cdot \left(1 + \frac{r-1}{r(b_1 - 1)} \right)^{b_1 - 1} \\ &\leq 1.05r \cdot \sqrt{\frac{k}{(b_1 - 1) \cdot b_2 \cdot \dots \cdot b_r}} \left(\frac{1}{\sqrt{2\pi}} \right)^{r-1} \cdot \frac{\left(1 + \frac{1}{r(b_1 - 1)} \right)^{(b_1 - 1)(r-1)}}{\prod_{i=2}^r \left(1 + \frac{1}{rb_i} \right)^{b_i}} \\ &\leq 1.05r \cdot \sqrt{\frac{k}{(b_1 - 1) \cdot b_2 \cdot \dots \cdot b_r}} \left(\frac{1}{\sqrt{2\pi}} \right)^{r-1}, \end{aligned}$$

where the third inequality holds because $1 + kx \leq (1 + x)^k$ and $(1 + x)(1 - x) = 1 - x^2 \leq 1$ for $x \geq 0$ and $k \geq 1$ and the last inequality holds because $h_2(x) = (1 + \frac{1}{x})^x$ is increasing for $x > 0$, implying that $(1 + \frac{1}{r(b_i - 1)})^{b_i - 1} \leq (1 + \frac{1}{rb_i})^{b_i}$ for all $2 \leq i \leq r$.

Noticing that $k + 1 = b_1 + b_2 + \dots + b_r$, $r \geq 3$ and $b_1, b_2, \dots, b_r \geq 2$, we have $\frac{k}{(b_1 - 1) \cdot b_2 \cdot \dots \cdot b_r} \leq \frac{5}{4}$ and it reaches the maximum value if and only if $r = 3, k = 5$ and $b_1 = b_2 = b_3 = 2$. Hence

$$\tilde{g}_r(\vec{\mathbf{b}}) \leq 1.05 \cdot \sqrt{\frac{5}{4}} \cdot r \left(\frac{1}{\sqrt{2\pi}} \right)^{r-1}.$$

Let $h_3(r) = r \left(\frac{1}{\sqrt{2\pi}} \right)^{r-1}$. Then

$$\frac{h_3(r+1)}{h_3(r)} = \frac{r+1}{\sqrt{2\pi} \cdot r} \leq 1 \text{ for } r \geq 3.$$

This implies that $h_3(r)$ is decreasing for $r \geq 3$. Hence we have

$$\tilde{g}_r(\vec{\mathbf{b}}) \leq 1.05 \cdot \sqrt{\frac{5}{4}} \cdot 3 \cdot \left(\frac{1}{2\pi} \right) \leq 0.561 < 0.5904 \leq f(k).$$

This completes the proof of Lemma 16. □

B Proof of Lemma 17

In this section, we give a detailed proof to show $f(k) > g_r(k)$ if one of the following cases holds:

$$(I). r = 2 \text{ and } k \geq 20; \quad (II). r \geq 3 \text{ and } k \geq 10; \quad (III). r \geq 6 \text{ and } 3 \leq k \leq 9.$$

In the remainder of this section, we establish Lemma 17 by dividing it into three lemmas, addressing cases (I), (II), and (III) respectively. In each lemma, we demonstrate that $g_r(k) < f_0(k) \leq f(k)$.

We will use Propositions 18 and 19 to illustrate the following numerical conclusions.

Lemma 20. *Suppose that $r = 2$ and $k \geq 20$ are positive integers. Then $g_r(k) < f(k)$.*

Proof. For $r = 2$, by Lemma 14 we have

$$g_2(k) = \max_{\substack{0 \leq a \leq b, \\ a+b=k-1}} g_r(a, b) = \max_{\substack{0 \leq a \leq b, \\ a+b=k-1}} \binom{k-1}{a} \left(\frac{2a+1}{2k} \right)^a \left(\frac{2b+1}{2k} \right)^b \left(1 + \frac{a(2b+1)}{(2a+1)(b+1)} \right).$$

For $k \geq 20$, using Proposition 18 (c), we have $f(k) \geq f_0(k) = 1 - (1 - \frac{1}{k})^{k-1} \geq 1 - 0.378 = 0.622$. So it suffices to show that under the conditions $k \geq 20$, $a \leq b$ and $a + b = k - 1$, we have $g_2(a, b) < 0.622$. To see this, by Proposition 18 (c), we see that under the assumption $k \geq 20$,

$$\text{if } a = 0, \quad g_2(a, b) = g_2(0, k-1) = \left(\frac{2k-1}{2k} \right)^{k-1} \leq 0.619 < 0.622, \text{ and}$$

$$\text{if } a = 1, \quad g_2(a, b) = g_2(1, k-2) = \frac{3(k-1)}{2k} \left(\frac{2k-3}{2k} \right)^{k-2} \left(1 + \frac{2k-3}{3(k-1)} \right) \leq \frac{3}{2} \cdot 0.246 \cdot \frac{5}{3} < 0.622.$$

Now it remains to consider when $2 \leq a \leq b$. Using Proposition 18 (c) and Proposition 19, we have

$$\begin{aligned} g_2(a, b) &= \binom{k-1}{a} \left(\frac{2a}{2(k-1)} \right)^a \left(1 + \frac{1}{2a} \right)^a \left(\frac{2b}{2(k-1)} \right)^b \left(1 + \frac{1}{2b} \right)^b \left(1 - \frac{1}{k} \right)^{k-1} \cdot \left(1 + \frac{a(2b+1)}{(2a+1)(b+1)} \right) \\ &\leq \frac{1.01 \left(\frac{k-1}{e} \right)^{k-1} \sqrt{2\pi(k-1)}}{\left(\frac{a}{e} \right)^a \sqrt{2\pi a} \cdot \left(\frac{b}{e} \right)^b \sqrt{2\pi b}} \cdot \frac{a^a}{(k-1)^a} \cdot \exp\left(\frac{a}{2a}\right) \cdot \frac{b^b}{(k-1)^b} \cdot \exp\left(\frac{b}{2b}\right) \cdot 0.378 \cdot \left(1 + \frac{2a}{2a+1} \right) \\ &= 1.01 \cdot 0.378 \cdot e \cdot \frac{(k-1)^{k-1+\frac{1}{2}}}{(k-1)^{a+b}} \cdot \frac{a^a}{a^{a+\frac{1}{2}}} \cdot \frac{b^b}{b^{b+\frac{1}{2}}} \cdot (2\pi)^{-\frac{1}{2}} \cdot \frac{4a+1}{2a+1} \\ &= 1.01 \cdot 0.378 \cdot e \cdot \sqrt{\frac{k-1}{2\pi ab}} \cdot \frac{4a+1}{2a+1}, \end{aligned}$$

where the last equality holds because $k - 1 = a + b$. If $a = 2$, then this implies that

$$g_2(a, b) \leq \sqrt{\frac{1}{2\pi}} \cdot 1.01 \cdot 0.378 \cdot \sqrt{\frac{k-1}{2(k-3)}} \cdot e \cdot \frac{9}{5} \leq \sqrt{\frac{1}{2\pi}} \cdot 1.01 \cdot 0.378 \cdot \sqrt{\frac{19}{34}} \cdot e \cdot \frac{9}{5} \leq 0.558 < 0.622.$$

So we may assume that $a \geq 3$. In this case, we have

$$g_2(a, b) \leq \sqrt{\frac{1}{2\pi}} \cdot 1.01 \cdot 0.378 \cdot \sqrt{\frac{k-1}{3(k-4)}} \cdot e \cdot 2 \leq \sqrt{\frac{1}{2\pi}} \cdot 1.01 \cdot 0.378 \cdot \sqrt{\frac{19}{48}} \cdot e \cdot 2 \leq 0.521 < 0.622.$$

Now for all integers $0 \leq a \leq b$ and $a + b = k - 1$, we establish $g_2(a, b) < f(k)$, finishing the proof. \square

Next we prove the case (II) of Lemma 17.

Lemma 21. *Suppose that $r \geq 3$ and $k \geq 10$ are positive integers. Then $g_r(k) < f(k)$.*

Proof. By Proposition 18, we have $f(k) \geq 1 - 1.06e^{-1} \geq 0.61$ for $k \geq 10$. So it suffices to show that for all integers $0 \leq a_1 \leq \dots \leq a_r$ with $a_1 + \dots + a_r = k - 1$, we have

$$g_r(a_1, \dots, a_r) = \binom{k-1}{a_1, \dots, a_r} \cdot \prod_{i=1}^r \left(\frac{ra_i + 1}{rk} \right)^{a_i} \cdot \left(1 + \frac{a_1}{ra_1 + 1} \sum_{i=2}^r \frac{ra_i + 1}{a_i + 1} \right) < 1 - 1.06e^{-1}.$$

First we consider there exists some $1 \leq \ell \leq r - 1$ such that $a_1 = \dots = a_\ell = 0$ and $a_{\ell+1}, \dots, a_r > 0$. Note that $\prod_{i=\ell+1}^r a_i$ is minimized by $(a_{\ell+1}, \dots, a_{r-1}, a_r) = (1, \dots, 1, k - r + \ell)$. Since $a_1 = 0$, we have

$$\begin{aligned} g_r(a_1, \dots, a_r) &= \binom{k-1}{a_1, \dots, a_r} \prod_{i=\ell+1}^r \left[\left(\frac{ra_i}{r(k-1)} \right)^{a_i} \left(1 + \frac{1}{ra_i} \right)^{a_i} \right] \cdot \left(\frac{r(k-1)}{rk} \right)^{k-1} \\ &\leq \frac{1.01 \left(\frac{k-1}{e} \right)^{k-1} \sqrt{2\pi(k-1)}}{\prod_{i=\ell+1}^r \left(\frac{a_i}{e} \right)^{a_i} \sqrt{2\pi a_i}} \cdot \prod_{i=\ell+1}^r \left[\frac{a_i^{a_i}}{(k-1)^{a_i}} \exp \left(\frac{a_i}{ra_i} \right) \right] \cdot 1.06 \cdot e^{-1} \\ &= (1.01 \cdot 1.06) \cdot \sqrt{\frac{k-1}{\prod_{i=\ell+1}^r a_i}} \cdot \left(\frac{1}{2\pi} \right)^{\frac{r-\ell-1}{2}} \cdot \exp \left(-\frac{\ell}{r} \right) \\ &\leq (1.01 \cdot 1.06) \cdot \sqrt{\frac{k-1}{(k-1) - (r-\ell-1)}} \cdot \left(\frac{1}{2\pi} \right)^{\frac{r-\ell-1}{2}} \cdot \exp \left(-\frac{\ell}{r} \right) \\ &\leq (1.01 \cdot 1.06) \cdot \exp \left(-\frac{r-\ell-1}{r} \right) \exp \left(-\frac{\ell}{r} \right) \\ &= 1.01 \cdot 1.06 \cdot \exp \left(-\frac{r-1}{r} \right) \leq 1.01 \cdot 1.06 \cdot \exp \left(-\frac{2}{3} \right) < 0.55 < 1 - 1.06e^{-1}, \end{aligned}$$

where the first inequality holds because of Proposition 18 (b) and Proposition 19, the second inequality holds since $\prod_{i=\ell+1}^r a_i \geq k - r + \ell$, and the third inequality holds because of Proposition 18 (d) (by taking $m = r - \ell - 1 \leq r - 2$).¹³ Hence, $g_r(a_1, \dots, a_r) < f(k)$ whenever $a_1 = 0$.

Now we suppose that $a_1 > 0$. Then $r \leq k - 1$ and $1 + \sum_{i=2}^r \frac{a_i}{ra_i + 1} \cdot \frac{ra_i + 1}{a_i + 1} \leq r$. In this case,

$$g_r(a_1, \dots, a_r) \leq r \binom{k-1}{a_1, \dots, a_r} \cdot \prod_{i=1}^r \left[\left(\frac{ra_i}{r(k-1)} \right)^{a_i} \left(1 + \frac{1}{ra_i} \right)^{a_i} \right] \cdot \left(\frac{r(k-1)}{rk} \right)^{k-1}.$$

Following the same discussion as above (e.g., taking $\ell = 0$), we have

$$g_r(a_1, \dots, a_r) \leq (1.01 \cdot 1.06 \cdot r) \cdot \sqrt{\frac{k-1}{k-r}} \cdot \left(\frac{1}{2\pi} \right)^{\frac{r-1}{2}}. \quad (21)$$

If $r = 3$, then $g_r(a_1, \dots, a_r) \leq (1.01 \cdot 1.06) \cdot \sqrt{\frac{9}{10-3}} \cdot \frac{1}{2\pi} \cdot 3 < 0.58 < 1 - 1.06e^{-1} \leq f(k)$. If $r \geq 4$, then

$$g_r(a_1, \dots, a_r) \leq (1.01 \cdot 1.06 \cdot r) \cdot \sqrt{\frac{k-1}{k-r}} \cdot \left(\frac{1}{2\pi} \right)^{\frac{r-1}{2}} \leq 1.01 \cdot 1.06 \cdot r^{\frac{3}{2}} \cdot \left(\frac{1}{2\pi} \right)^{\frac{r-1}{2}}, \quad (22)$$

¹³Note that if $r - \ell - 1 = 0$, then the third inequality holds trivially.

where we use $\sqrt{\frac{k-1}{k-r}} \leq \sqrt{r}$ for $1 \leq r \leq k-1$. Let $h(r) = r^{\frac{3}{2}} \left(\frac{1}{2\pi}\right)^{\frac{r-1}{2}}$. Then for all $r \geq 4$,

$$\frac{h(r+1)}{h(r)} = \left(\frac{r+1}{r}\right)^{\frac{3}{2}} \sqrt{\frac{1}{2\pi}} \leq \left(\frac{5}{4}\right)^{\frac{3}{2}} \sqrt{\frac{1}{2\pi}} \leq 1, \quad (23)$$

implying that $h(r)$ is decreasing. Then for $r \geq 4$, we can derive from (22) that

$$g_r(a_1, \dots, a_r) \leq 1.01 \cdot 1.06 \cdot r^{\frac{3}{2}} \cdot \left(\frac{1}{2\pi}\right)^{\frac{r-1}{2}} \leq 1.01 \cdot 1.06 \cdot 4^{\frac{3}{2}} \cdot \left(\frac{1}{2\pi}\right)^{\frac{3}{2}} \leq 0.55 < 1 - 1.06e^{-1} \leq f(k),$$

Now we have showed that $g_r(k) < f(k)$ for $r \geq 3$ and $k \geq 10$, completing the proof. \square

Finally, we prove the last case (III) of Lemma 17, whose proof is a modification of the case (II).

Lemma 22. *Suppose that $r \geq 6$ and $k \geq 3$ are positive integers. Then $g_r(k) < f(k)$.*

Proof. Let $r \geq 6$ and $k \geq 3$. By Proposition 18 (a), we have $(1 - \frac{1}{k})^{k-1} \leq \frac{4}{9} \leq 1.21 \cdot e^{-1}$ and $f(k) \geq f_0(k) \geq \frac{5}{9} \geq 0.555$. Also by Proposition 19, we have $\frac{(k-1)!}{\sqrt{2\pi(k-1)} \left(\frac{k-1}{e}\right)^{k-1}} \leq 1.05$. If there exists some $1 \leq \ell \leq r-1$ such that $a_1 = \dots = a_\ell = 0$ and $a_{\ell+1} > 0$, then following the same discussion as in the proof of Lemma 21, we can derive that

$$g_r(a_1, \dots, a_r) \leq 1.05 \cdot 1.21 \cdot \exp\left(-\frac{r-1}{r}\right) \leq 1.05 \cdot 1.21 \cdot \exp\left(-\frac{5}{6}\right) \leq 0.553 < \frac{5}{9} \leq f(k). \quad (24)$$

Now we may assume that $a_1 > 0$. Then following the same arguments as for (22) and (23), we derive

$$g_r(a_1, \dots, a_r) \leq 1.05 \cdot 1.21 \cdot r^{\frac{3}{2}} \cdot \left(\frac{1}{2\pi}\right)^{\frac{r-1}{2}} \leq 1.05 \cdot 1.21 \cdot 6^{\frac{3}{2}} \cdot \left(\frac{1}{2\pi}\right)^{\frac{5}{2}} \leq 0.190 < \frac{5}{9} \leq f(k),$$

where we use the fact that $h(r) = r^{\frac{3}{2}} \left(\frac{1}{2\pi}\right)^{\frac{r-1}{2}}$ is decreasing for $r \geq 6$. Putting everything together, we see that $g_r(k) < f(k)$ holds whenever $r \geq 6$ and $k \geq 3$. \square

We have completed the proof of Lemma 17. \square

C Numerical support for Table 1 and Table 2

Here we provide numerical support for Table 1 and Table 2.

To achieve this, we present Mathematica Codes along with outcomes that can be downloaded from the following link: <http://staff.ustc.edu.cn/~jiema/Color-biasPM-AppendixB.pdf>