ON ERDŐS PROBLEM #1034

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Abstract. We disprove a conjecture of Erdős and Faudree, which asked whether every graph G on n vertices with more than $n^2/4$ edges contains a triangle T with at least $(\frac{1}{2} - o(1))n$ vertices adjacent to at least two vertices of T. We construct, for every $\varepsilon > 0$ and all large n, a graph on n vertices with $e(G) > \frac{n^2}{4}$ in which every triangle has at most $(2 - \sqrt{5/2} + \varepsilon)n \leq (0.4189 + \varepsilon)n$ such vertices.

1. Introduction

In [2, p. 344], Erdős and Faudree posed the following conjecture (see also Problem #1034 in [1]):

Conjecture 1.1. *Let G be a graph on n vertices with more than $n^2/4$ edges. Then there exists a triangle T in G and vertices $y_1, \ldots, y_t$, where $t > (\frac{1}{2} - o(1))n$, such that every y_i is adjacent to at least two vertices of T.*

In this note we disprove this conjecture by constructing graphs with more than $n^2/4$ edges in which every triangle has at most $(2 - \sqrt{5/2} + o(1))n$ vertices adjacent to at least two of its vertices.

2. Counterexample Construction

Theorem 2.1. *For every $\varepsilon > 0$ and all sufficiently large integers n, there exists a graph G on n vertices with $e(G) > \frac{n^2}{4}$ such that for every triangle $T \subseteq G$,*

$$\left|\{\, v \in V(G) : v \text{ is adjacent to at least two vertices of } T \,\}\right| \leq (2 - \sqrt{5/2} + \varepsilon)\, n.$$

Proof. We present a two-parameter construction and then optimize the parameters.

Let $1/2 \leq \alpha \leq 1$ and let $s = s(n) \in \mathbb{N}$ be an integer to be specified later. Partition the vertex set as a disjoint union $V = B \cup S$ with

$$|B| = \lfloor \alpha n \rfloor, \qquad |S| = n - |B|.$$

Place all edges between B and S (so B–S induces a complete bipartite graph). Make S independent (no edges inside S). Inside B, partition the vertices into pairwise disjoint cliques of size s (and at most one residual part of size $< s$); add all edges inside each clique and no edges between distinct cliques of B.

Now, in this construction, every triangle $T \subseteq G$ is of exactly one of the following two types:

(i) $|T \cap B| = 2$ and $|T \cap S| = 1$; moreover the two vertices of $T \cap B$ lie in the same clique of B;

(ii) $|T \cap B| = 3$ and these three vertices lie in the same clique of B.

If $K(T)$ denotes the unique clique of B that contains $T \cap B$, then the set

$$Y(T) := \{\, v \in V(G) : v \text{ is adjacent to at least two vertices of } T \,\}$$

satisfies

$$Y(T) = S \;\cup\; K(T).$$

In particular,

$$|Y(T)| = |S| + |K(T)| \le |S| + s. \tag{2.1}$$

With the above construction, write $|B| = qs + r$ with $q = \lfloor |B|/s \rfloor$ and $0 \le r < s$. Then

$$e(B) = q\binom{s}{2} + \binom{r}{2} = \frac{|B|}{s}\binom{s}{2} + \left(\binom{r}{2} - \frac{r}{s}\binom{s}{2}\right) = \frac{|B|}{2}(s-1) \;-\; \frac{r}{2}(s-r).$$

Hence

$$e(B) \;\ge\; \frac{|B|}{2}(s-1) \;-\; \frac{s^2}{8}. \tag{2.2}$$

Set $b := |B| = \lfloor \alpha n \rfloor$ and write $b = \alpha n - \vartheta$ with $0 \le \vartheta < 1$. Then, since $\alpha \ge 1/2$, we have

$$|B||S| = b(n-b) = \alpha(1-\alpha)n^2 + (2\alpha - 1)\,\vartheta\, n - \vartheta^2 \;\ge\; \alpha(1-\alpha)n^2 - 1,$$

So by (2.2)

$$e(G) = e(B) + |B||S| \ge \alpha(1-\alpha)n^2 + \frac{|B|}{2}(s-1) - \frac{s^2}{8} - 1.$$

Using $|B| = \alpha n \pm O(1)$ and dividing by n^2, with $c := s/n$ and ignoring lower-order $o(1)$ terms, a sufficient condition for $e(G) > \frac{n^2}{4}$ is

$$\alpha(1-\alpha) \;+\; \frac{\alpha}{2}\,c \;-\; \frac{1}{8}\,c^2 > \frac{1}{4}. \tag{2.3}$$

Rewriting (2.3) gives the quadratic inequality

$$c^2 - 4\alpha c + 8(\alpha - \tfrac{1}{2})^2 < 0.$$

Thus the feasible c lie in the interval

$$c \in \big(c_1(\alpha),\, c_2(\alpha)\big), \qquad c_1(\alpha) := 2\alpha - \sqrt{2 - 4(\alpha-1)^2}\,, \quad c_2(\alpha) := 2\alpha + \sqrt{2 - 4(\alpha-1)^2}\,.$$

By (2.1), for every triangle T,

$$|Y(T)| \le |S| + s \le (1-\alpha)n + s + 1,$$

hence

$$\frac{|Y(T)|}{n} \;\le\; (1-\alpha) + c + o(1).$$

To minimize the right-hand side under the constraint (2.3), we choose

$$s := \lceil c_1(\alpha)\, n \rceil, \qquad c = \frac{s}{n} = \frac{\lceil c_1(\alpha) n \rceil}{n}.$$

Then

$$\frac{|Y(T)|}{n} \;\le\; (1-\alpha) + c_1(\alpha) \;+\; \frac{1}{n} \;+\; o(1).$$

Now we set

$$\Phi(\alpha) \;:=\; (1-\alpha) + c_1(\alpha) \;=\; 1 + \alpha - \sqrt{2 - 4(\alpha-1)^2}\,.$$

A routine calculation shows that Φ attains its minimum on $\alpha \in [1/2, 1]$ at

$$\alpha^* \;=\; 1 - \frac{1}{\sqrt{10}},$$

where

$$\min_{\alpha} \Phi(\alpha) \;=\; \Phi(\alpha^*) \;=\; 2 - \sqrt{5/2} \;=\; 0.418861\ldots$$

Therefore, choosing $\alpha = \alpha^*$ and the above s yields, for all triangles T,

$$|Y(T)| \;\le\; (2 - \sqrt{5/2} + o(1))\, n. \qquad \square$$

3. Further directions

In the original source of this problem [2, p. 344], Erdős wrote the following passage:

> *In a forthcoming paper of Faudree and myself, the following stronger conjecture is stated: In every* $G(n; \lfloor n^2/4 \rfloor + 1)$ *there is a triangle* (x_1, x_2, x_3) *so that there are at least* $\frac{n}{2}$ *and other vertices* $y_1, \ldots, y_t$*, with* $t > \frac{n}{2} - o(1)$*, each of which are joined to at least two of the x's. Perhaps this conjecture is a bit too optimistic, but if it is not true one should try to determine the largest* $h(n)$ *for which in every* $G(n; \lfloor n^2/4 \rfloor + 1)$ *there is a triangle* (x_1, x_2, x_3) *and* $h(n)$ *other vertices which are joined to at least two of the x's.*

Thus Erdős explicitly asked not only whether the "$\frac{n}{2}$–conjecture" is valid, but also what the optimal constant in $h(n)$ should be.

Combining our construction with the classical result on the existence of a book of size $n/6$ in every graph with $\lfloor n^2/4 \rfloor + 1$ edges, we obtain

$$\left(\frac{1}{6} - o(1)\right) n \leq h(n) \leq (2 - \sqrt{5/2} + o(1))n.$$

Determining the exact asymptotic constant

$$c_* := \lim_{n \to \infty} \frac{h(n)}{n}$$

remains open.

References

[1] T. F. Bloom, Erdős Problem #1034, `https://www.erdosproblems.com/1034`, accessed 2025-10-20.
[2] P. Erdős, Some of my favorite solved and unsolved problems in graph theory, *Quaestiones Math.* (1993), 333–350.

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