Towards a conjecture of Birmelé-Bondy-Reed on the Erdős-Pósa property of long cycles

Jie Ma* Chunlei Zu†

Abstract

A conjecture of Birmelé, Bondy and Reed states that for any integer $\ell \geq 3$, every graph G without two vertex-disjoint cycles of length at least ℓ contains a set of at most ℓ vertices which meets all cycles of length at least ℓ. They showed the existence of such a set of at most $2\ell+3$ vertices. This was improved by Meierling, Rautenbach and Sasse to $5\ell/3+29/2$. Here we present a proof showing that at most $3\ell/2+7/2$ vertices suffice.

1 Introduction

Let $\mathscr{F}$ be a family of graphs. For a given graph G, a subset X of $V(G)$ is called a *transversal* of $\mathscr{F}$ if the graph $G - X$ contains no member of $\mathscr{F}$. We say that $\mathscr{F}$ has the *Erdős-Pósa property*, if there is a function $f : \mathbb{N} \to \mathbb{N}$ such that for every positive integer k, every graph contains either k vertex-disjoint members of $\mathscr{F}$ or a transversal of $\mathscr{F}$ of size at most $f(k)$. A celebrated result of Erdős and Pósa [6] in 1965 states that the family of all cycles has the Erdős-Pósa property. Since then it has stimulated a new field of extensive research.

For any integer $\ell \geq 3$, let $\mathscr{F}_\ell$ denote the family of cycles of length at least ℓ. In 2007, Birmelé, Bondy and Reed [2] first proved that for every ℓ, $\mathscr{F}_\ell$ has the Erdős-Pósa property. To be precise, they showed that any graph without k vertex-disjoint cycles in $\mathscr{F}_\ell$ has a transversal of $\mathscr{F}_\ell$ of size at most $O(\ell k^2)$. The bound of the transversal was improved by Fiorini and Herinckx [7] to $O(\ell k \log k)$. In 2017, Mousset, Noever, Škorić and Weissenberger [12] further improved this to $O(\ell k + k \log k)$ and they also provided examples, showing that this is optimal up to the constant factor.

The present paper focuses on the base case $k = 2$ of the above problem, namely, considering graphs without two vertex-disjoint cycles in $\mathscr{F}_\ell$. As remarked by Birmelé, Bondy and Reed [2], the case $k = 2$ is "of particular importance". Indeed, all proofs of the above papers use inductive arguments. Birmelé, Bondy and Reed [2] made the following conjecture.

*School of Mathematical Sciences, University of Science and Technology of China, Hefei, Anhui 230026, China. Email: jiema@ustc.edu.cn. Research supported by the National Key R and D Program of China 2020YFA0713100, National Natural Science Foundation of China grant 12125106, and Anhui Initiative in Quantum Information Technologies grant AHY150200.

†School of CyberScience and School of Mathematical Sciences, University of Science and Technology of China, Hefei, Anhui 230026, China. Email: zucle@mail.ustc.edu.cn

Conjecture 1.1 (Birmelé, Bondy, and Reed [2]). *Let $\ell \geq 3$ and let G be a graph containing no two vertex-disjoint cycles of $\mathscr{F}_\ell$. Then there exists a transversal of $\mathscr{F}_\ell$ of size at most ℓ.*

Note that in view of the complete graph on $2\ell - 1$ vertices, the conjectured bound would be best possible. An early result of Lovász [10] implied the case $\ell = 3$. Birmelé [1] confirmed the cases $\ell \in \{4, 5\}$. For general ℓ, Birmelé, Bondy and Reed [2] proved that there exists a transversal of $\mathscr{F}_\ell$ of size at most $2\ell+3$. Later, Meierling, Rautenbach and Sasse [11] improved this to $5\ell/3 + 29/2$. Our main result here gives a further improvement as follows.

Theorem 1.2. *Let $\ell \geq 3$ be an integer. Let G be a graph containing no two vertex-disjoint cycles of $\mathscr{F}_\ell$. Then there exists a transversal of $\mathscr{F}_\ell$ of size at most $3\ell/2 + 7/2$.*

For more references on the Erdős-Pósa property, we would like to direct interested readers to the survey of Raymond and Thilikos [13] and [3–5, 8, 9, 14] for some recent developments (by no mean of a comprehensive list). The rest of the paper is organized as follows. In Section 2 we give the notation, while Section 3 is devoted to the proof of Theorem 1.2.

2 The notation

All graphs considered in this paper are finite, undirected and simple. Let X and Y be subgraphs of a graph G. For a vertex x in $V(G)$, we will use the notation $x \in X$ instead of $x \in V(X)$. An (X, Y)-*path* is a path in G which starts at a vertex of X and ends at a vertex of Y such that no internal vertex is contained in $V(X) \cup V(Y)$. Here we allow the possibility that $X = Y$. Let P be a path. By the *length* of P, we mean the number of edges in P. If x and y are two vertices of P, then xPy denotes the subpath of P with initial vertex x and terminal vertex y. We will reserve the term *disjoint* for *vertex-disjoint*.

Let C be a cycle with a prescribed orientation. For two vertices $x, y \in V(C)$, the *segment* xCy denotes the unique subpath of C from x to y following the orientation of C. So xCy and yCx are edge-disjoint whose union forms the cycle C. Consider two disjoint (C, C)-paths P and P' such that P is between u and v and P' is between u' and v'. We say that P and P' are *parallel* (with respect to C) if u, u', v', v appear in the given cyclic order on C and *crossing* (with respect to C) otherwise (see Figure 1).

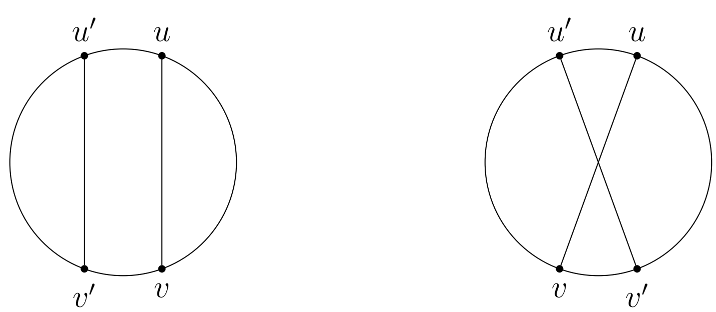


Figure 1. parallel and crossing paths

3 Proof of Theorem 1.2

Throughout the rest of this paper, let $\ell \geq 3$ be fixed. A cycle is called **long** if it has length at least ℓ (i.e., a cycle in $\mathscr{F}_\ell$) and **short** otherwise. We will assume by default that the orientation of any cycle is counterclockwise in all presentations and figures below.

Consider any graph G which contains no two disjoint long cycles. Our goal is to show that there exists a transversal of $\mathscr{F}_\ell$ of size at most $(3\ell+7)/2$ in G.

We now choose two long cycles C and D in G for the coming proof. Let C be a shortest long cycle of G with length L. It is clear that C intersects every long cycle of G, thus $V(C)$ is a transversal of $\mathscr{F}_\ell$. If $L \leq (3\ell+7)/2$, then the result follows. So we may assume that

$$L > (3\ell+7)/2.$$

We may also assume that there are at least $(3\ell+7)/2 \geq 8$ long cycles in G (as otherwise, there is a transversal of $\mathscr{F}_\ell$ of size at most $(3\ell+7)/2$ by taking a vertex from each long cycle). For every long cycle D of G other than C, let C_D denote a shortest segment of C containing all vertices in $V(C) \cap V(D)$. Note that $1 \leq |V(C_D)| \leq L$. Choose a long cycle D such that $|V(C_D)|$ is minimum. With respect to the given orientation of C, we let x and y be the first and last vertices of C_D, respectively. Clearly, $x, y \in V(C) \cap V(D)$.

The rest of the proof will be divided into two cases depending on whether $x = y$ or not. In each case, using Menger's theorem, we will find either two disjoint long cycles or a transversal of $\mathscr{F}_\ell$ of size at most $(3\ell+7)/2$, thereby finishing the proof of Theorem 1.2.

3.1 The case when $x \neq y$

Let X_1 be the set of $\lceil \ell/2 \rceil - 1$ vertices of C immediately preceding x, and let X_2 be the set of $\lceil \ell/2 \rceil - 1$ vertices of C immediately following y. Let $B = C \setminus (X_1 \cup X_2 \cup V(C_D))$.

We may assume that $G - (X_1 \cup X_2 \cup \{x, y\})$ contains some long cycle (as otherwise, $X_1 \cup X_2 \cup \{x, y\}$ is a transversal of $\mathscr{F}_\ell$ of size at most $2\lceil \ell/2 \rceil \leq \ell + 1$). Hence every long cycle D' in $G - (X_1 \cup X_2 \cup \{x, y\})$ intersects B by the minimality of C_D. Let $x_{D'} C_{D'} y_{D'}$ be a shortest segment of C containing $V(B) \cap V(D')$. From now on, choose a long cycle D' such that $|V(x_{D'} C x)|$ is minimum.

Let X_3 be the set of $\lceil \ell/2 \rceil - 1$ vertices of C immediately preceding $x_{D'}$. Clearly, X_1, X_2 and X_3 are pairwise disjoint. Otherwise, $X_1 \cup X_2 \cup X_3 \cup \{x, y, x_{D'}\}$ is a transversal of $\mathscr{F}_\ell$. Since $|X_1 \cup X_2 \cup X_3 \cup \{x, y, x_{D'}\}| \leq \sum_{i=1}^{3} |X_i| + 3 = 3\lceil \ell/2 \rceil \leq 3(\ell+1)/2$, we obtain the desired result. We know that $B \setminus X_3$ consists of two segments of C, say E_1 and E_2. One is adjacent to X_1 and another is adjacent to X_2 on C. Without loss of generality, we assume that E_1 is adjacent to X_1 and E_2 is adjacent to X_2 on C. Note that it is possible that $V(E_1)$ or $V(E_2)$ is empty.

Note that $C \setminus (X_1 \cup X_2 \cup X_3)$ consists of three segments of C, namely E_1, E_2 and E_3 (where $E_3 := C_D$). A (C, C)-path P with two endpoints x_0 and y_0 is called a **special** path between E_i and E_j, if $x_0 \in V(E_i)$, $y_0 \in V(E_j)$ and $i \neq j \in [3]$.

Claim A1. *Every special path has length at least $\ell - 1$.*

Proof. Let P be a special path between two vertices x_0 and y_0 of C. Let L_P be the length of P. Assume by symmetry that $x_0 \in V(E_1)$ and $y_0 \in V(E_2)$. Since $x_0 C y_0$ has length at least

$\ell-1$, $x_0Cy_0 \cup y_0Px_0$ forms a long cycle. By the minimality of C, $L_P \geq |X_3|+1 = \lceil \ell/2 \rceil$. Since y_0Cx_0 has length at least $\lceil \ell/2 \rceil$, we have that $y_0Cx_0 \cup x_0Py_0$ is also a long cycle. Thus the length of P is at least the length of x_0Cy_0, that is $L_P \geq |X_1|+|X_2|+1 = \lceil \ell/2 \rceil \times 2 - 1 \geq \ell - 1$, as desired. □

By the choice of D', we see D' is disjoint from $X_1 \cup X_2 \cup X_3 \cup V(E_2) \cup \{x, y\}$. Note that D' intersects D. It follows that there exists a $(E_1, D \setminus \{x, y\})$-path sQ_1t in $G - (X_1 \cup X_2 \cup X_3 \cup V(E_2) \cup \{x, y\})$, where $s \in V(E_1)$ and $t \in V(D) \setminus \{x, y\}$.

We may assume that there is still a long cycle D'' in $G - (X_1 \cup X_2 \cup X_3 \cup \{x, y, x_{D'}, t\})$. This is because that, otherwise, $X_1 \cup X_2 \cup X_3 \cup \{x, y, x_{D'}, t\}$ is a transversal of $\mathscr{F}_\ell$ of size at most $\sum_{i=1}^{3} |X_i| + 4 = \lceil \ell/2 \rceil \times 3 + 1 \leq 3\ell/2 + 5/2$. By the minimality of C_D and the choice of D', we know that D'' intersects E_2. Moreover, D'' intersects D. So there exists a $(E_2, D \setminus \{x, y, t\})$-path uQ_2v in $G - (X_1 \cup X_2 \cup X_3 \cup \{x, y, x_{D'}, t\})$, where $u \in V(E_2)$ and $v \in V(D) \setminus \{x, y, t\}$. We assert that $Q_2 \setminus \{u, v\}$ is disjoint from $C \cup sQ_1t$. Indeed, if not, then there is a special path between E_1 and E_2 from which it is easy to find a long cycle disjoint from D, a contradiction. Next, we show the following.

Claim A2. $v \in V(tDx) \setminus \{x, t\}$.

Proof. We have $v \in V(D) \setminus \{x, y, t\}$ and there are three segments of $D \setminus \{x, y, t\}$, namely $xDy \setminus \{x, y\}$, $yDt \setminus \{y, t\}$ and $tDx \setminus \{x, t\}$ (see Figure 2). Let $C_1 := sCx \cup tDx \cup sQ_1t$. Clearly, $tDx \cup sQ_1t$ contains a special path between E_1 and E_3. If $v \in V(xDy) \setminus \{x, y\}$, then $C_2 := yCu \cup uQ_2v \cup vDy$ and $uQ_2v \cup vDy$ contains a special path between E_2 and E_3, and if $v \in V(yDt) \setminus \{y, t\}$, then $C_2 := yCu \cup uQ_2v \cup yDv$ and $uQ_2v \cup yDv$ contains a special path between E_2 and E_3. By Claim A1, both C_1 and C_2 are long cycles. So in each case, we find two disjoint long cycles, a contradiction. □

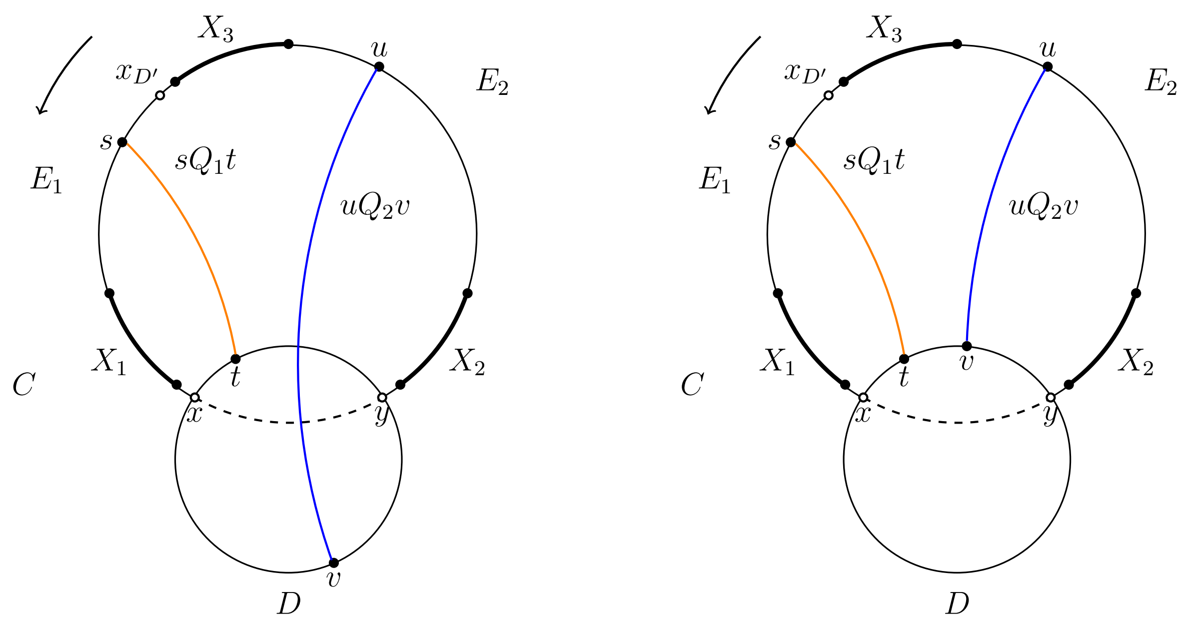


Figure 2. $v \in V(xDy) \setminus \{x, y\}$ and $v \in V(yDt) \setminus \{y, t\}$.

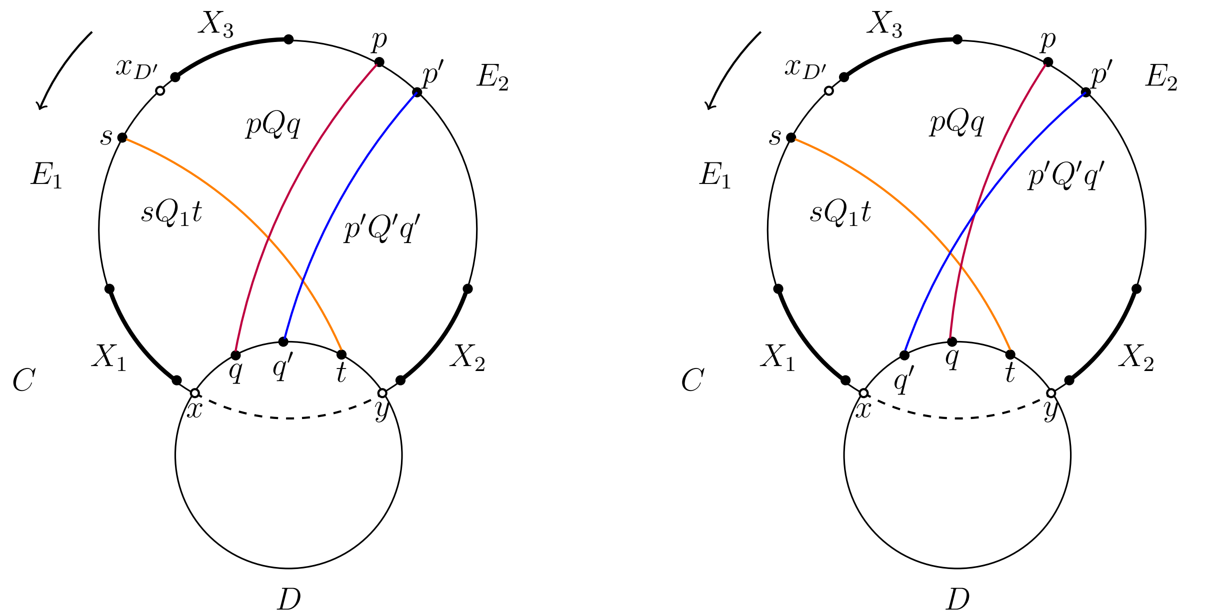


Figure 3. Two configurations in the proof of Claim A3.

Now, we see that uQ_2v is a $(E_2, tDx \setminus \{x,t\})$-path in $G - (X_1 \cup X_2 \cup X_3 \cup \{x, y, x_{D'}, t\})$ which has no internal vertex in $V(D \cup E_1 \cup sQ_1t)$.

Claim A3. *One cannot find two disjoint $(E_2, tDx \setminus \{x,t\})$-paths in $G - (X_1 \cup X_2 \cup X_3 \cup \{x, y, x_{D'}, t\})$ which has no internal vertex in $V(D \cup E_1 \cup sQ_1t)$.*

Proof. Suppose for a contradiction that such two paths exist, say pQq and $p'Q'q'$. There are two configurations as indicated in Figure 3. In the left configuration of the figure, we have two cycles $C_1 := pCx \cup pQq \cup qDx$ and $C_2 := yCp' \cup p'Q'q' \cup yDq'$. In the right side, we also have two cycles $C_1 := pCs \cup sQ_1t \cup pQq \cup tDq$ and $C_2 := yCp' \cup p'Q'q' \cup q'Dy$. Using Claim A1, we see that in both cases, C_1 and C_2 are two disjoint long cycles, a contradiction. $\square$

By Menger's theorem, Claim A3 shows that there is a vertex z meeting all $(E_2, tDx \setminus \{x,t\})$-paths in $G - (X_1 \cup X_2 \cup X_3 \cup \{x, y, x_{D'}, t\})$ which has no internal vertex in $V(D \cup E_1 \cup sQ_1t)$. Let $X := X_1 \cup X_2 \cup X_3 \cup \{x, y, z, x_{D'}, t\}$. Note that $|X| \le \sum_{i=1}^{3} |X_i| + 5 = 3\lceil \ell/2 \rceil + 2 \le (3\ell + 7)/2$. So it suffices to show that X is a transversal of $\mathscr{F}_\ell$. Suppose not. Then there is a long cycle D^* in $G - X$. Repeating the same proof as above, one can find a $(E_2, tDx \setminus \{x,t\})$-path in $G - X$ which has no internal vertex in $V(D \cup E_1 \cup sQ_1t)$, a contradiction to the definition of the vertex z. This completes the proof for the case $x \neq y$.

3.2 The case when $x = y$

In this case, clearly we may assume that $G - \{x\}$ contains at least one long cycle. Every long cycle in $G - \{x\}$ intersects each of the long cycles C and D. Thus there exists at least one (C, D)-path in $G - \{x\}$. We choose a (C, D)-path $y'P_0w$ in $G - \{x\}$, where $y' \in V(C)$ and $w \in V(D)$, such that the distance in C between x and y' is minimum. Without loss of generality, we assume that xCy' is a shortest path in C between x and y'.

Let X_1 be the set of $\lceil \ell/2 \rceil - 1$ vertices of C immediately preceding x, and let X_2 be the set of $\lceil \ell/2 \rceil - 1$ vertices of C immediately following y'. Let $A = xCy'$ and $B = C \setminus (X_1 \cup X_2 \cup V(xCy'))$. Since $|X_1 \cup X_2 \cup \{x, y', w\}| \leq \lceil 2\ell/2 \rceil + 1 \leq \ell + 2$, we may assume that there still is a long cycle D' in $G - (X_1 \cup X_2 \cup \{x, y', w\})$, which intersects both C and D. If $V(D' \cap C) \subseteq V(xCy')$, then by passing D', one can find a path from $V(xCy') \setminus \{x, y'\}$ to $V(D) \setminus \{x\}$ internally disjoint from $C \cup D$, a contradiction to the definition of wP_0y'. Therefore, every such cycle D' intersects B. Denote $x_{D'}C_{D'}y_{D'}$ to be a shortest segment of C containing $V(B) \cap V(D')$. From now on, choose a long cycle D' such that $|V(x_{D'}Cx)|$ is minimum. Let X_3 be the set of $\lceil \ell/2 \rceil - 1$ vertices of C immediately preceding $x_{D'}$. As before, we know that $B \setminus X_3$ consists of two segments of C, say E_1 and E_2. Without loss of generality, we assume that E_1 is adjacent to X_1 and E_2 is adjacent to X_2 on C. Let us call a (C, C)-path P with two endpoints x_0 and y_0 as a **special** path between E_i and E_j, if $x_0 \in V(E_i)$, $y_0 \in V(E_j)$ and $i \neq j \in [3]$. We point out that X_1, X_2 and X_3 are pairwise disjoint, and every special path has length at least $\ell - 1$.

By the choice of wP_0y', there is no $(A \setminus \{x, y'\}, D \setminus \{x, w\})$-path internally disjoint from C. It follows from the existence of D' that there is a $(E_1, D \setminus \{x, w\})$-path sQ_1t in $G - (X_1 \cup X_2 \cup X_3 \cup V(E_2) \cup \{x, y', w\})$, where $s \in V(E_1)$ and $t \in V(D) \setminus \{x, w\}$, internally disjoint from C, D and P_0.

Since $|X_1 \cup X_2 \cup X_3 \cup \{x, y', x_{D'}, w\}| \leq 3\lceil \ell/2 \rceil + 1 \leq 3\ell/2 + 5/2$, we may assume that there is a long cycle D'' in $G - (X_1 \cup X_2 \cup X_3 \cup \{x, y', x_{D'}, w\})$. By the choice of y' and $x_{D'}$, D'' intersects E_2 and $D \setminus \{x, w\}$. So there exists a $(E_2, D \setminus \{x, w\})$-path uQ_2v in $G - (X_1 \cup X_2 \cup X_3 \cup \{x, y', x_{D'}, w\})$, where $u \in V(E_2)$ and $v \in V(D) \setminus \{x, w\}$. We point out that uQ_2v has no internal vertex in $V(C \cup D \cup P_0 \cup (Q_1 \setminus \{t\}))$ (as otherwise, it is easy to find two disjoint long cycles in G as before; see Figure 4). Note that $D \setminus \{x, w\}$ consists of two segments of D, i.e., $xDw \setminus \{x, w\}$ and $wDx \setminus \{x, w\}$.

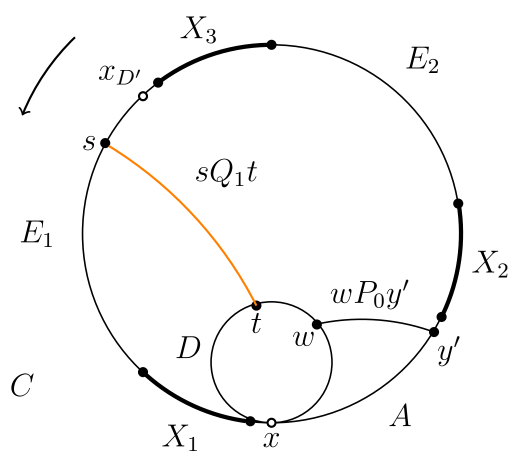
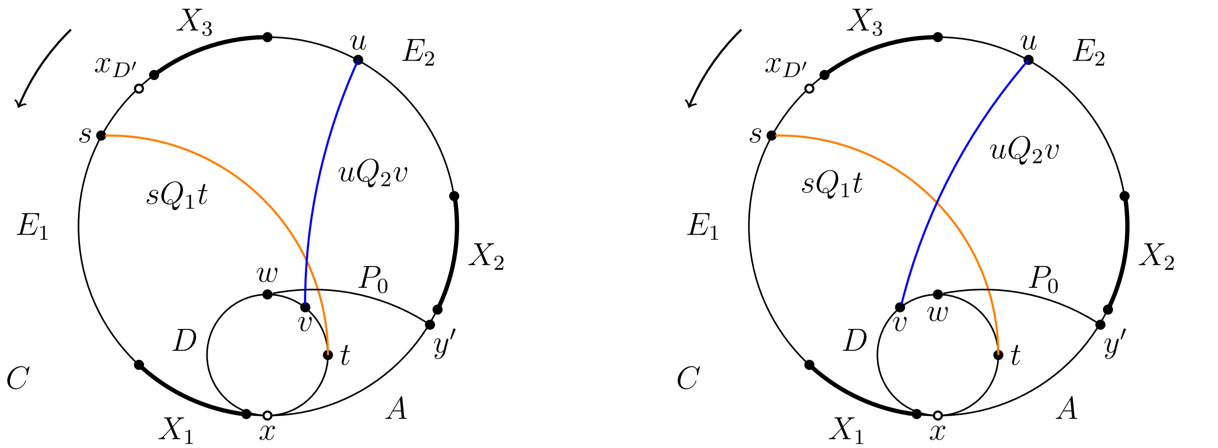


Figure 4. wP_0y'.

Claim B1. *If $t \in V(xDw) \setminus \{x, w\}$, then $v \in V(xDt) \setminus \{x\}$; if $t \in V(wDx) \setminus \{x, w\}$, then $v \in V(tDx) \setminus \{x\}$.*

Proof. First, consider $t \in V(xDw) \setminus \{x, w\}$ (see Figure 5). Suppose for a contradiction that $v \notin V(xDt) \setminus \{x\}$. Then either $v \in V(tDw) \setminus \{t, w\}$ or $v \in V(wDx) \setminus \{x, w\}$. Let

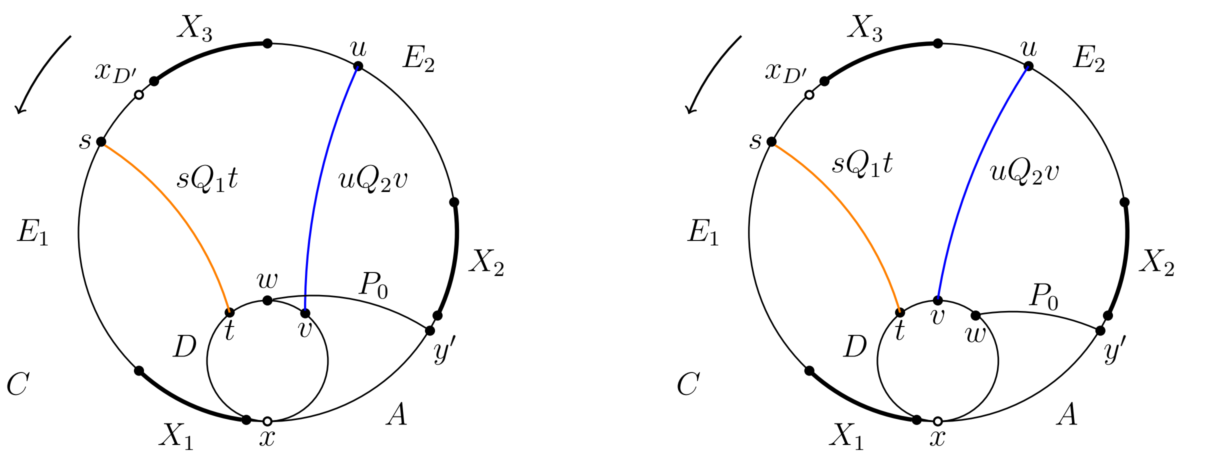
$C_1 := sCx \cup xDt \cup sQ_1t$. If $v \in V(tDw) \setminus \{t, w\}$, then define $C_2 := y'Cu \cup uQ_2v \cup vDw \cup y'P_0w$; otherwise $v \in V(wDx) \setminus \{x, w\}$, define $C_2 := y'Cu \cup uQ_2v \cup wDv \cup y'P_0w$. In both cases, C_1 and C_2 are disjoint long cycles, a contradiction.



$v \in V(tDw) \setminus \{t, w\}$ $\qquad$ $v \in V(wDt) \setminus \{t, w\}$

Figure 5. $t \in V(xDw) \setminus \{x, w\}$.

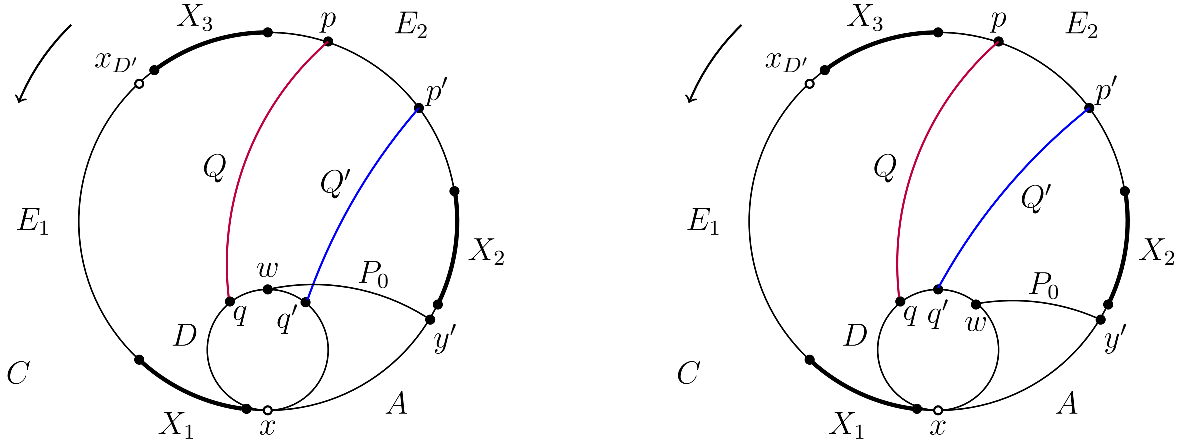
It remains to consider $t \in V(wDx) \setminus \{x, w\}$ (see Figure 6). Suppose that $v \notin V(tDx) \setminus \{x\}$. Then either $v \in V(xDw) \setminus \{x, w\}$ or $v \in V(wDt) \setminus \{w, t\}$. Let $C_3 := sCx \cup tDx \cup sQ_1t$. If $v \in V(xDw) \setminus \{x, w\}$, then define $C_4 := y'Cu \cup uQ_2v \cup vDw \cup y'P_0w$; otherwise $v \in V(wDt) \setminus \{w, t\}$, define $C_4 := y'Cu \cup uQ_2v \cup wDv \cup y'P_0w$. Again, in both cases, C_3 and C_4 are two disjoint long cycles, a contradiction. □



$v \in V(xDw) \setminus \{x, w\}$ $\qquad$ $v \in V(wDt) \setminus \{t, w\}$

Figure 6. $t \in V(wDx) \setminus \{x, w\}$.

Let pQq and $p'Q'q'$ be disjoint $(E_2, D\setminus\{x,w\})$-paths in $G-(X_1\cup X_2\cup X_3\cup\{x,y',x_{D'},w\})$, where $p,p'\in V(E_2)$ and $q,q'\in V(D)\setminus\{x,w\}$, such that they have no internal vertex in $V(C\cup D\cup P_0\cup(Q_1\setminus\{t\}))$. Without loss of generality, we assume that p' precedes p on C.

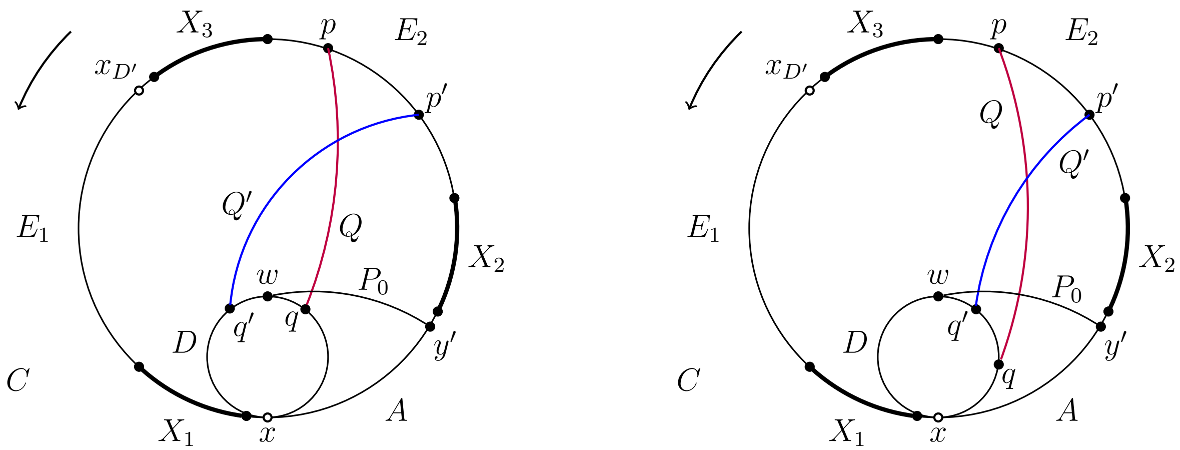


$q'\in V(xDw\setminus\{x,w\})$ $\qquad$ $q'\in V(wDx\setminus\{x,w\})$

Figure 7. pQq and $p'Q'q'$ are parallel.

Claim B2. *If pQq and $p'Q'q'$ are parallel, then $q,q'\in V(xDw)\setminus\{x,w\}$; if pQq and $p'Q'q'$ are crossing, then $q,q'\in V(wDx)\setminus\{x,w\}$;*

Proof. If pQq and $p'Q'q'$ are parallel (see Figure 7), then q' precedes q on D. Suppose for a contradiction that at least one of q and q' is not in $V(xDw)\setminus\{x,w\}$. Then $q\in V(wDx)\setminus\{x,w\}$. Let $C_1 := pCx\cup qDx\cup pQq$. If $q'\in V(xDw)\setminus\{x,w\}$, define $C_2 := y'Cp'\cup p'Q'q'\cup q'Dw\cup y'P_0w$; if $q'\in V(wDx)\setminus\{x,w\}$, define $C_2 := y'Cp'\cup p'Q'q'\cup wDq'\cup y'P_0w$. In both cases, C_1 and C_2 are disjoint long cycles, a contradiction.



$q'\in V(wDx)\setminus\{x,w\}$ $\qquad$ $q'\in V(xDw)\setminus\{x,w\}$

Figure 8. pQq and $p'Q'q'$ are crossing.

If pQq and $p'Q'q'$ are crossing (see Figure 8), then q precedes q' on D. Suppose for a contradiction that at least one of q and q' is not in $V(wDx) \setminus \{x,w\}$. Then $q \in V(xDw) \setminus \{x,w\}$. Let $C_3 := pCx \cup xDq \cup pQq$. If $q' \in V(wDx) \setminus \{x,w\}$, let $C_4 := y'Cp' \cup p'Q'q' \cup wDq' \cup y'P_0w$; otherwise $q' \in V(xDw) \setminus \{x,w\}$, let $C_4 := y'Cp' \cup p'Q'q' \cup q'Dw \cup y'P_0w$. In both cases, C_3 and C_4 are disjoint long cycles, a contradiction. □

Claim B3. *One cannot find two disjoint* $(E_2, D \setminus \{x,w\})$*-paths in* $G - (X_1 \cup X_2 \cup X_3 \cup \{x, y', x_{D'}, w\})$ *which has no internal vertex in* $V(C \cup D \cup P_0 \cup (Q_1 \setminus \{t\}))$.

Proof. Suppose for a contradiction that such paths exist, say pQq and $p'Q'q'$, where $p, p' \in V(E_2)$, $q, q' \in V(D) \setminus \{x,w\}$ and p' precedes p on C. By Claims B1 and B2, there are two configurations (see Figure 9). Be aware that there might be $q = t$. In the left configuration of Figure 9, pQq, $p'Q'q'$ and sQ_1t are pairwise crossing, and let $C_1 := pCs \cup sQ_1t \cup tDq \cup pQq$ and $C_2 := xCp' \cup p'Q'q' \cup q'Dx$. In the right configuration, pQq, $p'Q'q'$ and sQ_1t are pairwise parallel, and let $C_1 := pCs \cup sQ_1t \cup qDt \cup pQq$ and $C_2 := xCp' \cup p'Q'q' \cup xDq'$. It is easy to check that C_1 and C_2 are disjoint long cycles in both cases. □

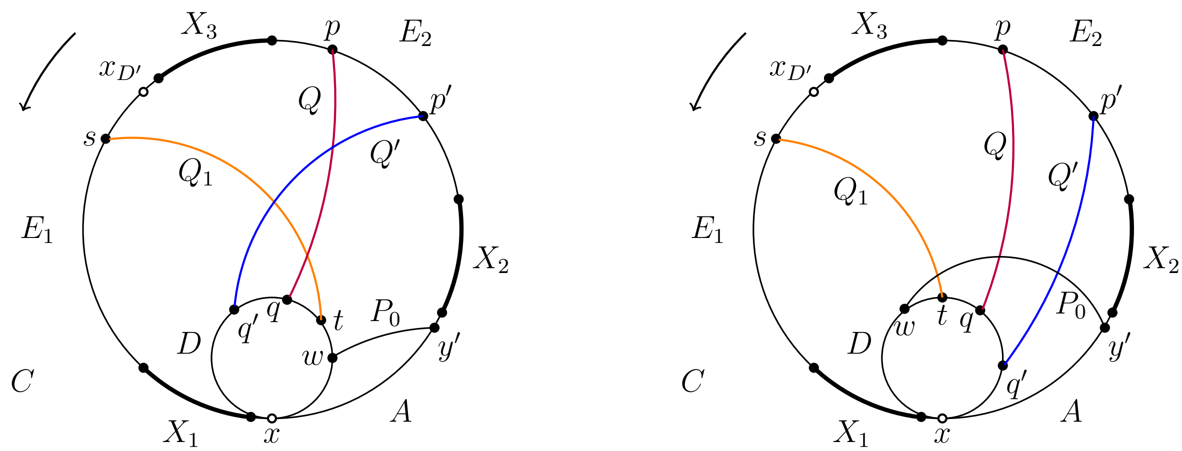


Figure 9. Two configurations in the proof of Claim B3.

By Menger's theorem, Claim B3 shows that there is a vertex z meeting all $(E_2, D \setminus \{x,w\})$-paths in $G - (X_1 \cup X_2 \cup X_3 \cup \{x, y', x_{D'}, w\})$ which has no internal vertex in $V(C \cup D \cup P_0 \cup (Q_1 \setminus \{t\}))$. Let $X := X_1 \cup X_2 \cup X_3 \cup \{x, y', z, x_{D'}, w\}$. Note that $|X| \leq \sum_{i=1}^{3} |X_i| + 5 = \lceil \ell/2 \rceil \times 3 + 2 \leq 3\ell/2 + 7/2$. So it is enough to show that X is a transversal of $\mathscr{F}_\ell$. Suppose not, then there is a long cycle D^* in $G - X$. As the same proof, one can show that there exists a $(E_2, D \setminus \{x,w\})$-path in $G - X$ which has no internal vertex in $V(C \cup D \cup P_0 \cup (Q_1 \setminus \{t\}))$. This is a contradiction to the definition of the vertex z. We have completed the proof of the case $x = y$ and thereby the proof of Theorem 1.2. □

References

[1] E. Birmelé, Thèse de doctorat, Université de Lyon 1, 2003.

[2] E. Birmelé, J. A. Bondy, and B. A. Reed, The Erdős-Pósa property for long circuits, Combinatorica 27 (2) (2007), 135–145.

[3] H. Bruhn, M. Heinlein, and F. Joos, Long cycles have the edge-Erdős-Pósa property, Combinatorica 39 (2019), no.1, 1–36.

[4] H. Bruhn, F. Joos, and O. Schaudt, Long cycles through prescribed vertices have the Erdős-Pósa property, J. Graph Theory 87 (2018), no.3, 275–284.

[5] W. Cames Van Batenburg, G. Joret, and A. Ulmer, Erdős-Pósa from ball packing, SIAM J. Discrete Math. 34 (2020), no.3, 1609–1619.

[6] P. Erdős and L. Pósa, On independent circuits contained in a graph, Canad. J. Math. 17 (1965), 347–352.

[7] S. Fiorini and A. Herinckx, A tighter Erdős-Pósa function for long cycles, J. Graph Theory 77 (2014), no.2, 111–116.

[8] M. Kang, O. Kwon, and M. Lee, Graphs without two vertex-disjoint S-cycles. Discrete Math. 343 (2020), no.10, 111997, 18 pp.

[9] E. Kim and O. Kwon, Erdős-Pósa property of chordless cycles and its applications. J. Combin. Theory Ser. B 145 (2020), 65–112.

[10] L. Lovász, On graphs not containing independent circuits (Hungarian), Mat. Lapok 16 (1965), 289–299.

[11] D. Meierling, D. Rautenbach, and T. Sasse, The Erdős-Pósa property for long circuits, J. Graph Theory 77 (2014), no.4, 251–259.

[12] F. Mousset, A. Noever, N. Škorić, and F. Weissenberger, A tight Erdős-Pósa function for long cycles, J. Combin. Theory Ser. B 125 (2017), 21–32.

[13] J.-F. Raymond and D. M. Thilikos, Recent techniques and results on the Erdős-Pósa property, Discrete Appl. Math. 231 (2017), 25–43.

[14] D. Weißauer, In absence of long chordless cycles, large tree-width becomes a local phenomenon. J. Combin. Theory Ser. B 139 (2019), 342–352.