# Extremal Combinatorics

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# 1 Turán's Theorem and Kövari-Sós-Turán Theorem

Let us begin this course by introducing some basic notations in graph theory. Let $G = (V, E)$ be a graph. The *degree* $d(v)$ of a vertex $v$ is the number of neighbors of $v$. Let $\Delta(G) := \max\{d(v)|v \in V\}$ be the *maximum degree* of $G$ and $\delta(G) := \min\{d(v)|v \in V\}$ be the *minimum degree*. The complete graph on $n$ vertices is denoted by $K_n$, while the complete $r$-partite graph with parts of sizes $n_1, n_2, ..., n_r$ is denoted by $K_{n_1,n_2,...,n_r}$.

Let $\mathcal{F}$ be a family of graphs. A graph $G$ is called $\mathcal{F}$-free if $G$ contains none of $\mathcal{F}$ as a subgraph. Let $\text{ex}(n, \mathcal{F})$ denote the largest possible number of edges in an $n$-vertex $\mathcal{F}$-free graph, and call it the *Turan number* or *extremal number* of $\mathcal{F}$.

## 1.1 Turán Density

If $\mathcal{F}$ be a family of graphs, the *Turán density* of $\mathcal{F}$ is denoted by $\pi(\mathcal{F}) = \lim_{n\to+\infty} \frac{\text{ex}(n,\mathcal{F})}{\binom{n}{2}}$. We can prove the following.

**Theorem 1.1.** *$\pi(\mathcal{F})$ exists for any family $\mathcal{F}$.*

*Proof.* Let $\pi_n = \text{ex}(n, \mathcal{F})/\binom{n}{2}$, then $\pi_n \in [0, 1]$. It suffices to show that $\{\pi_n\}$ is non-increasing. Let $G$ be an $n$-vertex $\mathcal{F}$-free graph with $\text{ex}(n, \mathcal{F})$ edges. By double-counting the number of pairs $(e, T)$ where $e \in G[T]$ and $T \subseteq \binom{G}{n-1}$, we can get

$$\#(e, T) = \sum_{e\in E(G)} \binom{n-2}{n-3} = (n-2)\text{ex}(n, \mathcal{F})$$

and

$$\#(e, T) = \sum_{T\subseteq\binom{V(G)}{n-1}} e(G[T]) \leq n \cdot \text{ex}(n-1, \mathcal{F}).$$

Together we have $(n-2)\text{ex}(n, \mathcal{F}) \leq n \cdot \text{ex}(n-1, \mathcal{F})$, implying that $\pi_n \leq \pi_{n-1}$, as desired. ∎

## 1.2 Mantel's Theorem

**Theorem 1.2** (Mantel)**.** *If $G$ is an $n$-vertex $K_3$-free graph, then $e(G) \leq \lfloor \frac{n^2}{4} \rfloor$ with equality if and only if $G = K_{\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil}$.*

*Proof.* We will prove it by induction on $n$. It is trivial for $n \leq 3$. So assume $n \geq 4$. By deleting some edges, we can also assume $e(G) = \lfloor \frac{n^2}{4} \rfloor$. Then there exists a vertex $v$ with $d(v) \leq \lfloor \frac{n}{2} \rfloor$. Let $G' = G - \{v\}$. Clearly $G'$ is an $(n-1)$-vertex $K_3$-free graph with $e(G') \geq \lfloor \frac{n^2}{4} \rfloor - \lfloor \frac{n}{2} \rfloor = \lfloor \frac{(n-1)^2}{4} \rfloor$. By induction, we know $G' = K_{\lfloor \frac{n-1}{2} \rfloor \lceil \frac{n-1}{2} \rceil}$ with two parts $A, B$. Also it is easy to see that $N(v)$ is a subset of $A$ or $B$. Otherwise, there exists a $K_3$ in $G$. Then one can verify that $N(v) = A$ or $N(v) = B$ and thus $G = K_{\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil}$. ∎

## 1.3 Turán's Theorem

Let *Turán graph* $T_r(n)$ be the complete balanced $r$-partite graph on $n \geq r$ vertices. That is $V = V_1 \cup V_2 ... \cup V_r$ and $|V_i| = \lfloor \frac{n}{r} \rfloor$ or $\lceil \frac{n}{r} \rceil$, such that all pairs in $V_i \times V_j$ form edges.

Before proving the Turán's Theorem, let us show three easy observations on $T_r(n)$ as follows.

(i) $e(T_r(n)) = \sum_{0\le i<j<r} \lfloor \frac{n+i}{r} \rfloor \lfloor \frac{n+j}{r} \rfloor$ achieves the unique maximum in all $n$-vertex $r$-partite graphs.

(ii) $T_r(n-1) = T_r(n) - \{v\}$, where $d(v) = \delta(T_r(n)) = n - \lceil \frac{n}{r} \rceil$.

(iii) $T_r(n)$ has the highest minimum degree among all $n$-vertex graphs with the same number of edges.

Next, we will give two different proofs based on above observations.

**Theorem 1.3** (Tuŕan). *Let $G$ be an $n$-vertex $K_{r+1}$-free graph. Then $e(G) \le e(T_r(n))$ with equality holds if and only if $G = T_r(n)$.*

*Proof.* (first): We prove it by induction on $n$. The base case $n = r$ is clear. Let $n \ge r+1$. By observation (iii), there exists a vertex $v$ with $d(v) \le \delta(T_r(n))$. Let $G' = G - \{v\}$. We see $e(G') = e(G) - d(v) \ge e(T_r(n)) - \delta(T_r(n)) = e(T_r(n-1))$. By induction, we know $G' = T_r(n-1)$. Then we claim that $G$ is a $r$-partite graph. As otherwise, each part of $G'$ contains a neighbor of $v$, implying that these $r$ vertices together with $v$ form a $K_{r+1}$. Hence by (i) we get $G = T_r(n)$. ∎

*Proof.* (second): Let us prove it by induction on $r$. It is clear that Mantel's Theorem gives the case for $r = 2$. Assume $r \ge 3$. Let $u \in V(G)$ with $d(u) = \Delta(G)$. Let $S = N(u)$ and $T = V \backslash S$. We see $G[S]$ is $K_r$-free. Now let $G'$ be obtained from $G$ by deleting all edges in $G[T]$ and adding all missing edges in $(S, T)$. Then we see $G'$ is $K_{r+1}$-free. And the number of missing edges in $(S, T)$ is $|S||T| - e_G(S, T)$. Now we claim that $e(G') = e(G)$ and $e(G[T]) = 0$. Since

$$2e(G[T]) + e_G(S, T) = \sum_{x\in T} d_G(x) \le \Delta(G)|T| = |S||T|,$$

we have

$$e(G') = e(G) - e(G[T]) + (|S||T| - e_G(S, T)) \ge e(G) + e(G[T]).$$

This confirms the claim. Clearly $G = G'$. We see $G[S]$ is $K_r$-free and contain the maximum number of edges on $|S|$ vertices. By induction we know $G[S]$ must be $(r-1)$-partite. Thus $G$ is $r$-partite. Finally, by (i) we get $G = T_r(n)$. ∎

Next, we consider the Turán number of complete bipartite graphs.

### 1.4 Kövari-Sós-Turán Theorem

**Theorem 1.4** ( Kövari-Sós-Turán). *For any integers $t \ge s \ge 2$, we have*

$$\text{ex}(n, K_{s,t}) \le \frac{1}{2}(t-1)^{\frac{1}{s}} n^{2-\frac{1}{s}} + \frac{1}{2}(s-1)n.$$

*Proof.* Let $G$ be any $n$-vertex $K_{s,t}$-free graph. Let the number of stars $K_{1,s}$ in $G$ be $T$. On the one hand, for a fixed vertex $v$, there are $\binom{d_G(v)}{s}$ many $K_{1,s}$ rooted at it. Then $T = \sum_{v\in V(G)} \binom{d_G(v)}{s}$. Here we define

$$\binom{x}{s} = \begin{cases} \frac{x(x-1)\cdots(x-s+1)}{s!}, & x \ge s \\ 0, & \text{otherwise.} \end{cases}$$

On the other hand, for any fixed $s$ vertices, there are at most $t-1$ vertices which are adjacent to all these $s$ vertices. Thus we have $T \le (t-1)\binom{n}{s}$.

Combining them and using Jensen's inequality, we get

$$(t-1)\frac{n^s}{s!} \ge (t-1)\binom{n}{s} \ge \sum_{v\in V(G)} \binom{d_G(v)}{s} \ge n\binom{\sum_{v\in V(G)} d_G(v)/n}{s} \ge n \cdot \frac{(2e(G)/n - s + 1)^s}{s!}.$$

Thus we have $e(G) \le \frac{1}{2}(t-1)^{\frac{1}{s}} n^{2-\frac{1}{s}} + \frac{1}{2}(s-1)n$. ∎

These theorems also tell us that $\pi(K_{r+1}) = 1 - \frac{1}{r}$ and $\pi(K_{s,t}) = 0$ for any integers $r, s, t \ge 1$.

# 2 Hypergraph KST, Supersaturation Lemma

## 2.1 Erdös-Moon Theorem

**Theorem 2.1** (Erdös-Moon). *Let $G$ be an $n$-vertex graph with more than $\frac{1}{2}s^{1+1/s}n^{2-1/s} + 2sn$ edges. Then it has at least $\Omega(p^{s^2}n^{2s})$ copies of $K_{s,s}$, where $p = e(G)/\binom{n}{2}$ is the edge-density of $G$.*

*Proof.* Let $M$ denote the number of stars $K_{1,s}$ in $G$. We see $M = \sum_{v\in V}\binom{d(v)}{s}$. For a subset $S \subseteq V(G)$ of size $s$, let $f(S)$ be the number of vertices adjacent to all vertices of $S$. Then we see $M = \sum_{S\in\binom{V}{s}} f(S)$. And $G$ has $\frac{1}{2}\sum_{S\in\binom{V}{s}}\binom{f(S)}{s}$ copies of $K_{s,s}$. By Jensen's inequality, we also get

$$\frac{1}{2}\sum_{S\in\binom{V}{s}}\binom{f(S)}{s} \ge \frac{1}{2}\binom{n}{s}\binom{\sum_{S\in\binom{V}{s}} f(S)/\binom{n}{s}}{s} \ge \frac{1}{2}\binom{n}{s}\binom{M/\binom{n}{s}}{s} = \Omega(n^{s-s^2})M^s$$

$$= \Omega(n^{s-s^2})\left(\sum_{v\in V}\binom{d(v)}{s}\right)^s \ge \Omega(n^{s-s^2})\left(n\binom{\sum_{v\in V} d(v)/n}{s}\right)^s = \Omega(p^{s^2}n^{2s}).$$

∎

## 2.2 Hypergraph Kövari-Sós-Turán Theorem

Let $K^{(k)}_{(t_1,t_2,\ldots,t_k)}$ be the complete $k$-partite $k$-graph. For convenience, we denote $K_{t:k} = K^{(k)}_{(t,t,\ldots,t)}$. For a $k$-graph $G$ and $v \in V(G)$, the *link-hypergraph* $G_v$ is a $(k-1)$-graph on the vertex set $V(G) - \{v\}$ where $e \in E(G_v)$ if and only if $e \cup \{v\} \in E(G)$.

**Theorem 2.2** (Erdős). *Let $k, t \ge 2$ be integers. Then there exists a constant $c = c(k,t)$ such that any $k$-graph $G$ with $e(G) = p\binom{n}{k} \ge cn^{k-(1/t)^{k-1}}$ has at least $\Omega(p^{t^k}n^{tk})$ copies of $K_{t:k}$.*

*Proof.* We prove this by induction on $k$. It is clear that the Erdös-Moon theorem gives the case for $k = 2$. We may assume it holds for $k-1$ with $k \ge 3$. Suppose $G$ is a $k$-graph with $cn^{k-(1/t)^{k-1}}$ edges. Let $V_1 = \{v \in V(G) : d(v) \ge cn^{(k-1)-(1/t)^{k-2}}\}$ and $V_2 = V(G) \setminus V_1$. Then $\sum_{v\in V_2} d(v) \le n \cdot cn^{(k-1)-(1/t)^{k-2}} \ll e(G)$, implying that $\sum_{v\in V_1} d(v) = (k - o(1))e(G)$.

For each $v \in V_1$, the link-hypergraph $G_v$ is a $(k-1)$-graph with $e(G_v) = d(v)$. By induction, $G_v$ has at least $\Omega\left(\left(\frac{e(G_v)}{\binom{n-1}{k-1}}\right)^{t^{k-1}} \cdot (n-1)^{t(k-1)}\right) = \Omega(n^{t(k-1)-(k-1)t^{k-1}})d(v)^{t^{k-1}}$ copies of $K_{t:(k-1)}$.

Let $\vec{S} = (S_1, ..., S_{k-1})$, where $|S_i| = t$. We denote by $f(\vec{S})$ the number of vertices $v$ such that $v \cup S_1 \cup \cdots \cup S_{k-1}$ induces a copy of $K^{(k)}_{(1,t,...,t)}$ in $G$. Clearly each copy of $K^{(k)}_{(1,t,...,t)}$ in $G$ associates with a $K_{t:(k-1)}$ in a unique $G_v$ for $v \in V$. Thus the number of $K^{(k)}_{(1,t,...,t)}$ in $G$ is

$$\sum_{\vec{S}} f(\vec{S}) \geq \Omega(n^{t(k-1)-(k-1)t^{k-1}}) \sum_{v \in V_1} d(v)^{t^{k-1}}.$$

We already have $\sum_{v \in V_1} d(v) = (k - o(1))e(G)$ and $|V_1| \leq n$. By Jensen's inequality, we can get

$$\sum_{\vec{S}} f(\vec{S}) \geq \Omega(n^{t(k-1)-(k-1)t^{k-1}}) \cdot n \left(\frac{d(v)}{n}\right)^{t^{k-1}} = \Omega(p^{t^{k-1}} n^{t(k-1)+1}).$$

Finally, the number of $K_{t:k}$ in $G$ is equal to

$$\frac{1}{k} \sum_{\vec{S}} \binom{f(\vec{S})}{t} \geq \Omega(n^{t(k-1)}) \left(\frac{\sum_{\vec{S}} f(\vec{S})}{n^{t(k-1)}}\right)^t \geq \Omega(n^{t(k-1)})(p^{t^{k-1}} n)^t = \Omega(p^{t^k} n^{tk}).$$

∎

This theorem gives us an upper bound for the extremal number of $K_{t:k}$ as follows, and also implies that $\pi(K_{t:k}) = 0$.

**Theorem 2.3** (Hypergraph Kövari-Sós-Turán ).

$$\mathrm{ex}_k(n, K_{t:k}) = O(n^{k-(1/t)^{k-1}}).$$

## 2.3 Supersaturation Lemma

**Theorem 2.4** (Supersaturation lemma). *Let $F$ be a $k$-graph with $k \geq 2$. For any $\epsilon > 0$, there exist positive constants $\delta = \delta(F, \epsilon)$ and $n_0 = n_0(F, \epsilon)$ such that for any $n$-vertex $k$-graph $G$ with $n > n_0$, if $G$ has at least $ex(n, F) + \epsilon \cdot n^k$ edges, then it contains at least $\delta n^v$ copies of $F$, where $v = |V(F)|$.*

*Proof.* By the definition of $\pi(F)$, we can find an integer $m$ such that $ex(m', F) < (\pi(F) + \frac{\epsilon}{2})\binom{m'}{k}$ for any $m' \geq m$. Let $n > n_0 \gg m$. Assume the $n$-vertex $k$-graph $G$ has $(\pi(F) + \epsilon)\binom{n}{k}$ edges. We use $T$ to denote the number of pairs $(e, M)$, where $M \in \binom{V(G)}{m}$ and $e \in G[M]$. On the one hand, we have

$$T = \sum_{e \in E(G)} \binom{n-k}{m-k} = e(G)\binom{n-k}{m-k} = (\pi(F) + \epsilon)\binom{n}{k}\binom{n-k}{m-k} = (\pi(F) + \epsilon)\binom{n}{m}\binom{m}{k}.$$

On the other hand, if we let $\mathcal{A} = \{M \in \binom{V(G)}{m} : e(G[M]) > (\pi(F) + \frac{\epsilon}{2})\binom{m}{k}\}$, then we get

$$T = \sum_{M \in \binom{V(G)}{m}} e(G[M]) = \sum_{M \in \mathcal{A}} e(G[M]) + \sum_{M \notin \mathcal{A}} e(G[M]) \leq |\mathcal{A}|\binom{m}{k} + (\binom{n}{m} - |\mathcal{A}|)(\pi(F) + \frac{\epsilon}{2})\binom{m}{k}.$$

The above two inequalities indicate that $(\pi(F) + \epsilon)\binom{n}{m} \leq |\mathcal{A}| + (\binom{n}{m} - |\mathcal{A}|)(\pi(F) + \frac{\epsilon}{2})$. So $|\mathcal{A}| \geq \frac{\epsilon}{2}\binom{n}{m}/(1 - \frac{\epsilon}{2} - \pi(F))$. Since each $M \in \mathcal{A}$ satisfies $e(G[M]) > (\pi(F) + \frac{\epsilon}{2})\binom{m}{k} > ex(m, F)$, we know $G[M]$ has at least one copy of $F$. As each $F$ can be contained in at most $\binom{n-v}{m-v}$ choices of $M \subset \mathcal{A}$, we finally get the number of $F$-copies in $G$ is at least $\frac{|\mathcal{A}|}{\binom{n-v}{m-v}} = \frac{\frac{\epsilon}{2}\binom{n}{m}}{(1-\frac{\epsilon}{2}-\pi(F))\binom{n-v}{m-v}} = \delta n^v$.

∎

# 3 Blowup Lemma and Erdős-Stone-Simonovits Theorem

## 3.1 Blowup Lemma

Given a $k$-graph $F$, the *$t$-blowup* $F(t)$ is a $k$-graph obtained from $F$ by replacing each $v \in V(F)$ by an independent subset $I(v)$ of size $t$ and replacing every edge $\{v_1, v_2, ..., v_k\} \in E(F)$ by a complete $k$-partite $k$-graph $I(v_1), I(v_2), ..., I(v_k)$. For example, the $t$-blowup of $K_r$ is $K_r(t) = T_r(rt)$.

**Theorem 3.1** (Blowup Lemma)**.** *For any $k$-graph $F$ and $t \geq 1$, we have $\pi(F(t)) = \pi(F)$. In other words,* $\text{ex}(n, F) \leq \text{ex}(n, F(t)) \leq \text{ex}(n, F) + \varepsilon n^k$ *for any $t, \varepsilon > 0$ and $n \geq n(t, \varepsilon)$.*

*Proof.* Let $f = |V(F)|$. First, clearly we have $\text{ex}(n, F) \leq \text{ex}(n, F(t))$ since $F \subseteq F(t)$. Then we can assume that for sufficiently large $n$, there exist $\varepsilon > 0$ and an $F(t)$-free $k$-graph $G$ on $n$ vertices, with $e(G) > \text{ex}(n, F) + \varepsilon n^k$. By supersaturation lemma, $G$ contains at least $\delta\binom{n}{f}$ copies of $F$ where $\delta = \delta(\varepsilon, F)$.

Next we define an auxiliary $f$-graph $G^*$ on $V(G)$ where $X \in \binom{n}{f}$ is an edge of $G^*$ if and only if $G[X]$ contains a copy of $F$. So $e(G^*) \geq \frac{\delta n^f}{f!} = \Omega(n^f)$. Take an integer $T$ such that $n \gg T \gg t, f$. As $e(G^*) = \Omega(n^f) > \text{ex}(n, K_{T:f})$, by hypergraph Kövari-Sós-Turán theorem, we see $G^*$ has at least one copy of $K_{T:f}$.

There are $f!$ possible ways to embed a copy of $F$ in an edge of $K_{T:f}$. Now we give $f!$ colors to the edges of $K_{T:f}$. Notice that one color stands for one possible embedding. By pigeonhole principle, there is a color whose number of edges is at least $\frac{T^f}{f!} > \text{ex}(vT, K_{t:f})$. So $K_{T:f}$ contains a monochromatic copy of $K_{t:f}$. This copy of $K_{t:f}$ gives a copy of $F(t)$ in $G$. ∎

The *chromatic number* of a graph $G$, denoted by $\chi(G)$, is the minimum integer $k$ such that $V(G)$ can be partitioned into $k$ independent sets.

**Fact 3.2.** $\chi(G) \leq k \Leftrightarrow G$ *is is $k$-partite.*

**Fact 3.3.** *From the blowup lemma, for any $m \geq 1$,*

$$\pi(T_r(rm)) = \pi(K_r) = 1 - \frac{1}{r-1}.$$

## 3.2 Erdős-Stone-Simonovits Theorem

**Theorem 3.4** (Erdős-Stone)**.** *If $\chi(F) = r$, then $\pi(F) = \pi(K_r) = 1 - \frac{1}{r-1}$.*

*Proof.* There exists an integer $m$ such that $F \subseteq T_r(rm)$. So $\text{ex}(n, F) \leq \text{ex}(n, T_r(rm))$, implying that $\pi(F) \leq \pi(T_r(rm)) = \pi(K_r) = 1 - \frac{1}{r-1}$. For the lower bound, since $\chi(F) = r$, we see $T_{r-1}(n)$ is $F$-free. Then $\text{ex}(n, F) \geq e(T_{r-1}(n))$. Combining them, we get

$$\pi(F) = \pi(K_r) = 1 - \frac{1}{r-1} = 1 - \frac{1}{\chi(F) - 1}$$

∎

Obviously, the above theorem has the following version.

**Theorem 3.5** (Erdős-Stone)**.** *For any graph $F$ and $n$,* $\text{ex}(n, F) = (1 - \frac{1}{\chi(F)-1} + o(1))\binom{n}{2}$ *where* $o(1) \to 0$ *as* $n \to +\infty$.

Let $\mathcal{F}$ be a family of graphs. Let the chromatic number $\chi(\mathcal{F}) = \min_{F\in\mathcal{F}} \chi(F)$.

**Theorem 3.6** (Erdős-Stone; observed by Simonovits)**.** *For any family $\mathcal{F}$ of graphs,*

$$\pi(\mathcal{F}) = 1 - \frac{1}{\chi(\mathcal{F}) - 1}.$$

Let us see some easy observations.

1. For any family $\mathcal{F}$ of graphs, $\pi(\mathcal{F}) \in \{0, \frac{1}{2}, \frac{2}{3}, ..., \frac{r-1}{r}, ...\}$.
2. For any graph $F$, $\mathrm{ex}(n, F) = (1 - \frac{1}{\chi(F)-1} + o(1))\binom{n}{2}$. When $\chi(F) = 2$(i.e. $F$ is bipartite), this becomes $\mathrm{ex}(n, F) = o(n^2)$.

The problem of finding $\mathrm{ex}(n, F)$ for bipartite graphs $F$ is call *degenerate Tuán problem.*

## 3.3 Quantitative Version of Erdős-Stone-Simonovits Theorem

Erdős-Stone-Simonovits theorem says $\mathrm{ex}(n, T_r(rm)) \le (1 - \frac{1}{r-1} + \varepsilon)\binom{n}{2}$. That means for fixed $m, \epsilon$, it holds for large $n$. Now we consider the counterpart that for fixed $n, \varepsilon$, how large $m$ can be?

For convenience, let us define a function

$$f_r(n, \varepsilon) = \max\left\{m : \mathrm{ex}(n, T_r(rm)) \le \mathrm{ex}(n, K_r) + \varepsilon\binom{n}{2} - 1\right\}.$$

For functions $f, g$, we write $f \lesssim g$ if $\lim_{n\to+\infty} \frac{f(n)}{g(n)} \le 1$ and $f \sim g$ if $\lim_{n\to+\infty} \frac{f(n)}{g(n)} = 1$.

The Erdős-Renyí random graph $G(n, p)$ for $0 \le p \le 1$ is a graph on $n$ vertices, where each pair of vertices forms an edge with probability $p$, independently at random. In particular, $G(n, \frac{1}{2})$ can be viewed as an equally distributed probability space which consists of all labeled $n$-vertex graphs.

**Theorem 3.7** (upper bound proved by Bollobás-Erdős)**.**

$$\log_{1/\varepsilon} n \lesssim f_2(n, \varepsilon) \lesssim 2\log_{1/\varepsilon} n.$$

*Proof.* First consider the lower bound. Let $m = \log_{1/\varepsilon} n$, so $n^{-1/m} = \varepsilon$. Thus

$$\mathrm{ex}(n, T_2(2m)) = \mathrm{ex}(n, K_{m,m}) \le \frac{1}{2}(m-1)^{1/m} n^{2-1/m} + \frac{1}{2}(m-1)n \lesssim \frac{\varepsilon n^2}{2}.$$

This proves $\log_{1/\varepsilon} n \lesssim f_2(n, \varepsilon)$.

Second, let $t = 2\log_{1/\varepsilon} n$. To show $f_2(n, \varepsilon) < t$, we need to prove $\mathrm{ex}(n, K_{t,t}) > \varepsilon\binom{n}{2} - 1$. It suffices to construct a $n$-vertex $K_{t,t}$-free graph $G$ with at least $\varepsilon\binom{n}{2} - 1$ edges. Consider Erdős-Renyí random graph $G(n, \varepsilon)$. Let $X$ be the number of $K_{t,t}$ in $G(n, \varepsilon)$. We have

$$\mathbb{E}[X] = \frac{1}{2}\binom{n}{2t}\binom{2t}{t}\varepsilon^{t^2} < n^{2t}\varepsilon^{t^2} = (n^2\varepsilon^t)^t.$$

Since $\varepsilon^t = \varepsilon^{2\log_{1/\varepsilon} n} = n^{-2}$, we see $\mathbb{E}[X] < 1$. By average, there exists a graph $G$ such that $e(G) - X \ge \mathbb{E}[e(G) - X] > \varepsilon\binom{n}{2} - 1$. Let $G'$ be obtained from $G$ by deleting one edge for each copy of $K_{t,t}$ in $G$. Then $G'$ is $K_{t,t}$-free with $e(G') \ge e(G) - X > \varepsilon\binom{n}{2} - 1$. ∎

The best general bound is used by Szemeredi's regularity lemma.

**Theorem 3.8** (Ishigami)**.** *For any* $r \geq 2$ *and* $\varepsilon = o(1)$*, we have* $f_r(n, \varepsilon) \sim f_2(n, \varepsilon)$*. Thus*

$$\log_{1/\varepsilon} n \lesssim f_r(n, \varepsilon) \lesssim 2 \log_{1/\varepsilon} n.$$

**Theorem 3.9.** *For any* $\varepsilon \in (0, \frac{1}{r(r+1)})$ *where* $r$ *is fixed,*

$$f_{r+1}(n, \varepsilon) \leq f_2 \left( \lceil \frac{n}{r} \rceil, r(r+1)\varepsilon \right).$$

*Proof.* Assume not. Let $t = f_2(\lceil \frac{n}{r} \rceil, r(r+1)\varepsilon) + 1$, then there exists a $K_{t,t}$-free $\lceil \frac{n}{r} \rceil$-vertex graph $H$ with $e(H) > r(r+1)\varepsilon \binom{\lceil \frac{n}{r} \rceil}{2} - 1$. Let $G$ be obtained from $T_r(n)$ by adding $H$ into a part of size $\lceil \frac{n}{r} \rceil$. We claim that $G$ is $T_{r+1}((r+1)t)$-free (prove it as an exercise). Thus we have

$$\text{ex}(n, T_{r+1}((r+1)t)) \geq e(G) = e(T_r(n)) + e(H) \geq \text{ex}(n, K_{r+1}) + \varepsilon \binom{n}{2}.$$

By definition, $f_{r+1}(n, \varepsilon) \leq t - 1 = f_2(\lceil \frac{n}{r} \rceil, r(r+1)\varepsilon)$. ∎

# 4 Bondy-Simonovits Theorem on Even Cycles

We consider the upper bound of $\mathrm{ex}(n, C_{2k})$ for $k \geq 2$ in this lecture.

**Theorem 4.1** (Bondy-Simonovits). *There is a constant $c > 0$ such that for any $k \geq 2$,*

$$\mathrm{ex}(n, C_{2k}) \leq ckn^{1+1/k}.$$

**Remark**: The original proof gives $c = 100$.

First, let us give some remarks. The current best bound on $\mathrm{ex}(n, C_{2k})$ is as follows.

**Theorem 4.2** (Bukh-Jiang, 2016).

$$\mathrm{ex}(n, C_{2k}) \leq 80\sqrt{k} \log k \cdot n^{1+1/k} + 10k^2 n.$$

Their proof heavily relies on $A$-$B$ path Lemma.

**Conjecture 4.3** (Erdős-Simonovits). *For $k \geq 2$,*

$$\mathrm{ex}(n, C_{2k}) = \Theta(n^{1+1/k}).$$

This conjecture is known for $k = 2, 3, 5$ only.

In the following lecture, we will give three different proofs of Theorem 4.1. Let us get into the first one by introducing the $A$-$B$ path lemma.

## 4.1 The First Proof

**Theorem 4.4** ($A$-$B$ path Lemma). *Let $H$ be a graph consisting of a cycle with a chord, and let $(A, B)$ be a non-trivial partition of $V(H)$. Then for any $\ell < |V(H)|$, there is an $(A, B)$-path of length $\ell$ in $H$, unless $\ell$ is even and $H$ is bipartite with the partition $(A, B)$.*

**The first proof of Theorem 4.1.** Let the cycle $C = (0, 1, \ldots, n-1, 0)$ with chord $(0, r)$. We take indices under modulo $n$. Denote $\chi : V(H) \to \{0, 1\}$ by $\chi(i) = 1$ for $i \in A$ and $\chi(i) = 0$ for $i \in B$. Let $P = \{p \in Z_n^+ : \chi(i) = \chi(i+p) \text{ holds for any } i\}$. So if $\ell \notin P$, we can find an $(A, B)$-path of length $\ell$ using only the edges of $C$.

It suffices for us to consider $\ell \in P$. Let $m \in P$ be the smallest positive integer in $P$. Then $m|n$ (exercise). For all $\ell$ with $m \nmid \ell$, there exists some $(A, B)$-path of length $\ell$.(By the definition of $m$.) So we only need to consider $\ell = km$.

**Case 1:** Suppose the chord $(0, r)$ satisfies that $1 < r \leq m$. Since $m \nmid (m + r - 1)$, there is some $-m < j \leq 0$ such that $\chi(j) \neq \chi(j + m + r - 1) = \chi(j + km + r - 1)$. Consider the path $(j, j+1, \ldots, 0, r, r+1, \ldots, j+m+r-1, \ldots, j+km+r-1)$. This is an $(A, B)$-path of length $km = \ell$.

**Case 2:** Suppose $m < r < n - m$. For $-m \leq j \leq 0$, we define 2 paths: $P_j = (j, j+1, \ldots, 0, r, r-1, \ldots, r-j-m+1)$ and $Q_j = (m+j, m+j-1, \ldots, 0, r, r+1 \ldots, r-j-1)$. We see both paths have length $m$.

**(i)** Suppose there is a $j$ with $-m \leq j \leq 0$ such that $P_j$ or $Q_j$ is an $(A, B)$-path. Then we can extend it to an $(A, B)$-path of length $km = \ell$ by adding a subpath of length $m$ at a time.

**(ii)** We may assume that $P_j$ and $Q_j$ are not $(A, B)$-paths for all $-m \leq j \leq 0$. Then we have $\chi(j) = \chi(r-j-m+1)$, $\chi(m+j) = \chi(r-j-1)$ for any $-m \leq j \leq 0$. So $\chi(r-j+1) = \chi(r-j-1)$, for any $-m \leq j \leq 0$. That is $\chi(i) = \chi(i+2)$ for any $i$. Then for $m = 2$, we have $2|n$ and the

vertices of $C$ alternate between $A$ and $B$. If the chord $(0, r)$ is in the same part, we can check that $H$ contains $A$-$B$ paths of all possible lengths. Otherwise, the chord $(0, r)$ is between $A$ and $B$, then $H$ is bipartite with the partition $(A, B)$.
**Case 3:** $n - m \leq r < n - 1$. This case is the same as **Case 1**. ∎

*Proof of Theorem 4.1.* We will show

$$\mathrm{ex}(n, C_{2k}) \leq 2kn^{1+1/k} + 6(k-1)n.$$

Let $G$ be an $n$-vertex $C_{2k}$-free graph with more than $2kn^{1+1/k} + 6(k-1)n$ edges. Then $G$ has a bipartite subgraph $H'$ with $e(H') > kn^{1+1/k} + 3(k-1)n$. Further, $H'$ contains a bipartite subgraph $H$ with $\delta(H) > kn^{1/k} + 3(k-1)$. Let $T$ be a breadth-first search tree (BFS tree) with root $x$ in $H$. Let $L_i = \{u \in V(H) : d_H(x, u) = i\}$ for $i \geq 1$. Since $H$ is bipartite, each $L_i$ is stable.

First we claim that $e(L_{i-1}, L_i) \leq (k-1)(|L_{i-1}| + |L_i|)$ for each $1 \leq i \leq k$. Suppose not, $e(L_{i-1}, L_i) > (k-1)(|L_{i-1}| + |L_i|)$ for some $i \geq 2$. Then $H(L_{i-1}, L_i)$ contains a subgraph $H_1$ with $\delta(H_1) \geq k$. Then $H_1$ has an even cycle $C$ of length at least $2k$ with a chord. Let $A = V(C) \cap L_{i-1}$ and $B = V(C) \cap L_i$. Let $T'$ be a subtree of $T$ such that $A \subseteq V(T')$ and subject to this, $T'$ is minimal. Let $y$ be the root of $T'$. As $T'$ is minimal, $y$ has at least 2 branches. Let $A'$ be the subset of $A$ formed by all vertices from one branch of $T'$. Then $A \setminus A' \neq \emptyset$. Let $B' = B \cup (A \setminus A')$. Then $(A', B')$ is not a bipartition of $H_1$. Let $\ell$ be the distance between $x$ and $y$. Then $\ell < i - 1$ and $2k - 2i + 2\ell + 2 < 2k \leq |V(C)|$. By $A$-$B$ path Lemma, we can find an $(A', B')$-path $P$ of length $2k - 2i + 2\ell + 2$ in $H_1$ between $a \in A'$ and $b \in B'$. As $|P|$ is even, $b \in A \setminus A'$. Let $P_a, P_b$ be the unique paths in $T'$ that connect $y$ to $a$ and $b$ respectively. Then $P \cup P_a \cup P_b$ is a cycle of length $2k$ in $H$, a contradiction.

Next we show that $|L_i| \geq n^{1/k}|L_{i-1}|$ for any $i \in [k]$. We prove this by induction on $i$. Base case $i = 1$ is trivial since $\delta(H) > kn^{1/k} + 3(k-1)$. For $i \geq 2$, we have

$$\begin{aligned}(kn^{1/k} + 3(k-1))|L_{i-1}| &\leq \sum_{v \in L_{i-1}} d_H(v) = e(L_{i-2}, L_{i-1}) + e(L_{i-1}, L_i) \\ &\leq (k-1)(|L_{i-2}| + 2|L_{i-1}| + |L_i|) \leq (k-1)(3|L_{i-1}| + |L_i|).\end{aligned}$$

So $|L_i| \geq \frac{kn^{1/k}}{k-1}|L_{i-1}| \geq n^{1/k}|L_{i-1}|$, as desired. Now we see $|L_k| \geq n$, a contradiction. ∎

## 4.2 The Second Proof

Next, we move into the second proof of Theorem 4.1.

**Lemma 4.5** (Lemma 2.6 in [1]). *Let $H$ be a connected graph where each edge is colored by color 1 or color 2. Suppose that there is at least one edge of each color. If the number of edges of color 1 is at least $(p+1)|V(H)|$, then there exists a path of length $p$ in $H$, whose first edge is colored by color 2 and all other edges are colored by color 1.*

*Proof.* Exercise. ∎

**The second proof of Theorem 4.1.** This is gave by Jiang-Ma in [1]. We aim to show

$$\mathrm{ex}(n, C_{2k}) \leq 8kn^{1+1/k} + 24kn.$$

Let $G$ be a $n$-vertex $C_{2k}$-free graph with more than $8kn^{1+1/k}+24kn$ edges. Then $G$ has a bipartite subgraph $H'$ with $e(H') > 4kn^{1+1/k} + 12kn$. Further, $H'$ contains a bipartite subgraph $H$ with $\delta(H) > 4kn^{1/k} + 12k$. Similarly, let $T$ be a breadth-first search tree (BFS tree) with root $x$ in $H$. Let $L_i = \{u \in V(H) | d_H(x,u) = i\}$ for $i \geq 1$. Since $H$ is bipartite, each $L_i$ is stable.

First we claim that $e(L_{i-1}, L_i) \leq 4k(|L_{i-1}| + |L_i|)$ for each $1 \leq i \leq k$. Suppose not, $e(L_{i-1}, L_i) > 4k(|L_{i-1}| + |L_i|)$ for some $i \geq 2$. Take a connected component $H^*$ with $d(H^*) \geq 8k$ in $H(L_{i-1}, L_i)$. Let $T'$ be a subtree of $T$ with $V(H^*) \cap L_{i-1} \subseteq V(T')$, and subject to this, $T'$ is minimal. Let $X$ be the subset of $V(H^*) \cap L_{i-1}$ which formed by all vertices from one branch of $T'$. Let $Y = (V(H^*) \cap L_{i-1}) \setminus X$. Color all edges in $H^*$ by color 1 if it has an end in $X$ and by color 2 if it has an end in $Y$. Then we can assume that the number of edges with color 1 is at least $2k|V(H^*)|$. By Lemma 4.5, there is a path $P$ of length at least $2k-1$ whose first edge is colored by color 2 and all other edges are colored by color 1. So we can find consecutive even cycles of length $2t+2, 2t+4, \ldots, 2t+2k-2$ where $t$ is the distance between $L_{i-1}$ and the root of $T'$. Since $t < i \leq k$, there is a cycle of length $2k$, a contradiction.

Next, we claim that $|L_i| \geq n^{1/k}|L_{i-1}|$ for any $i \in [k]$. We prove this by induction on $i$. Base case $i = 1$ holds as $\delta(H) > 4kn^{1/k} + 12k$. For $i \geq 2$, we have

$$\begin{aligned}(4kn^{1/k} + 12k)|L_{i-1}| &\leq \sum_{v \in L_{i-1}} d_H(v) = e(L_{i-2}, L_{i-1}) + e(L_{i-1}, L_i) \\ &\leq 4k(|L_{i-2}| + 2|L_{i-1}| + |L_i|) \leq 4k(3|L_{i-1}| + |L_i|),\end{aligned}$$

then $|L_i| \geq n^{1/k}|L_{i-1}|$. Finally, we get $|L_k| \geq n$, a contradiction.

∎

### 4.3 The Third Proof

It is due to Oliver Janzer in [2].

For graphs $H$ and $G$, we write $hom(H, G)$ for the number of homomorphisms $\phi : V(H) \to V(G)$ such that if $xy \in E(H)$ then $\phi(x)\phi(y) \in E(G)$.

**Definition 4.6.** *A homomorphic copy of $C_{2\ell}$ is a tuple $(x_1, ..., x_{2\ell}) \in V(G)^{2\ell}$ such that $x_1x_2, ..., x_{2\ell}x_1 \in E(G)$.*

**Lemma 4.7.** *Let $\ell \geq 2$ and $G$ be a properly edge-colored graph. Then the number of homomorphic copies of $C_{2\ell}$ which are NOT rainbow is at most $16\ell\Big(\ell\Delta(G)hom(C_{2\ell-2}, G)hom(C_{2\ell}, G)\Big)^{\frac{1}{2}}$.*

*Proof.* It suffices to prove that the number of $(x_1, ..., x_{2\ell})$ with $x_1x_2, x_2x_3, ..., x_{2\ell}x_1 \in E(G)$ such that $c(x_1x_2) = c(x_ix_{i+1})$ for some $2 \leq i \leq \ell + 1$ is at most

$$8l\Big(l\Delta(G)hom(C_{2\ell-2}, G)hom(C_{2\ell}, G)\Big)^{\frac{1}{2}}.$$

For $s \geq 1$, let $\alpha_s$ be the number of walks of length $\ell - 1$ in $G$ whose endpoints $y$ and $z$ satisfy

$$2^{s-1} \leq hom_{y,z}(P_{\ell-1}, G) \leq 2^s.$$

And let $\beta_s$ be the number of walks of length $\ell$ in $G$ whose endpoints $y$ and $z$ satisfy

$$2^{s-1} \leq hom_{y,z}(P_\ell, G) \leq 2^s.$$

Then we have

$$\sum_{s\geq 1} \alpha_s \cdot 2^{s-1} \leq hom(C_{2\ell-2}, G),$$

and

$$\sum_{s\geq 1} \beta_s \cdot 2^{s-1} \leq hom(C_{2\ell}, G).$$

For $s$, $t$, write $\gamma_{s,t}$ for the number of homomorphic copies $x_1x_2...x_{2\ell}x_1$ of $C_{2\ell}$ such that $c(x_1x_2) = c(x_ix_{i+1})$ for some $2 \leq i \leq \ell+1$, we have

$$2^{s-1} \leq hom_{x_1,x_{l+2}}(P_{\ell-1}, G) < 2^s,$$

and

$$2^{t-1} \leq hom_{x_2,x_{l+2}}(P_\ell, G) < 2^t,$$

which implies that

$$\gamma_{s,t} \leq \alpha_s \cdot \Delta(G) \cdot 2^t,$$

and

$$\gamma_{s,t} \leq \beta_t \cdot \ell \cdot 2^s.$$

We want to compute $\sum_{s,t\geq 1} \gamma_{s,t}$, let $q$ be the integer such that

$$\left(\frac{\ell \cdot hom(C_{2\ell}, G)}{\Delta(G) hom(C_{2\ell-2}, G)}\right)^{\frac{1}{2}} \leq 2^q < 2\left(\frac{\ell \cdot hom(C_{2\ell}, G)}{\Delta(G) hom(C_{2\ell-2}, G)}\right)^{\frac{1}{2}}.$$

Then, we have

$$\begin{aligned}\sum_{s\leq t-q} \gamma_{s,t} &\leq \ell \sum_{s\leq t-q} \beta_t 2^s \leq \ell \sum_{t\geq 1} \beta_t 2^{t-q+1} \leq \ell \cdot 2^{-q+2} hom(C_{2\ell}, G) \\ &\leq 4\big(\ell\Delta(G) hom(C_{2\ell-2}, G) hom(C_{2\ell}, G)\big)^{\frac{1}{2}},\end{aligned}$$

and

$$\begin{aligned}\sum_{s>t-q} \gamma_{s,t} &\leq \Delta(G) \sum_{s>t-q} 2^t \alpha_s \leq \Delta(G) \sum_{s\geq 1} 2^{s+q} \alpha_s \leq \Delta(G) 2^{q+1} hom(C_{2\ell-2}, G) \\ &\leq 4\big(\ell\Delta(G) hom(C_{2\ell-2}, G) hom(C_{2\ell}, G)\big)^{\frac{1}{2}}.\end{aligned}$$

Hence, we have

$$\sum_{s,t} \gamma_{s,t} \leq 8\ell\Big(\ell\Delta(G) hom(C_{2\ell-2}, G) hom(C_{2\ell}, G)\Big)^{\frac{1}{2}}.$$

∎

**Corollary 4.8.** *Let $k \geq 2$ be an integer and let $G$ be a properly edge-colored non-empty graph on $n$ vertices with $hom(C_{2k}, G) \geq 2^{8k} k^{3k} n \Delta(G)^k$. Then $G$ contains a rainbow cycle of length at most $2k$.*

*Proof.* Let $\ell$ be the smallest positive integer satisfying

$$hom(C_{2\ell}, G) \geq 2^{8\ell} k^{3\ell} n \Delta(G)^{\ell}.$$

Since

$$hom(C_2, G) = 2e(G) \leq n\Delta(G),$$

we have

$$2 \leq \ell \leq k.$$

Note that

$$hom(C_{2\ell-2}, G) < 2^{8(\ell-1)} k^{3(\ell-1)} n\Delta(G)^{\ell-1} \leq \frac{hom(C_{2\ell}, G)}{2^8 k^3 \Delta(G)} \leq \frac{hom(C_{2\ell}, G)}{2^8 \ell^3 \Delta(G)}.$$

By Lemma 4.7, the number of homomorphic $C_{2\ell}$ in $G$ which are NOT rainbow is at most

$$16\ell\Big(\ell\Delta(G) hom(C_{2\ell-2}, G) hom(C_{2\ell}, G)\Big)^{\frac{1}{2}} < hom(C_{2\ell}, G).$$

Hence, there is a homomorphic copy of $C_{2\ell}$ in $G$ which is a rainbow cycle. ∎

**Definition 4.9.** *An injectively homomorphic $C_{2\ell}$ is a homomorphic copy $(x_1, ..., x_{2\ell})$ of $C_{2\ell}$, such that all vertices $x_i$ are distinct.*

**Lemma 4.10.** *Let $\ell \geq 2$ be a positive integer and let $G$ be a graph. Then the number of homomorphic, but not injective copies of $C_{2\ell}$ in $G$ is at most*

$$16\ell\Big(\ell\Delta(G) hom(C_{2\ell-2}, G) hom(C_{2\ell}, G)\Big)^{\frac{1}{2}}.$$

*Proof.* The proof is almost identical to the proof of Lemma 4.7. The only difference is that instead of bounding those homomorphic copies $(x_1, ..., x_{2l})$ with $c(x_1x_2) = c(x_ix_{i+1})$ for some $2 \leq i \leq \ell + 1$. All details go through exactly the same way. ∎

**Lemma 4.11.** *Let $H$ be a bipartite graph and suppose that it does NOT contain a non-empty subgraph with minimum degree at least $d$. Then the largest eigenvalue of $H$ is at most $2\sqrt{d\Delta(H)}$.*

**Lemma 4.12.** *Let $H$ be a bipartite graph with parts $Y$ and $Z$. Suppose that $H$ does NOT contain a non-empty subgraph with minimum degree at least $d$. Then there exist bipartite graphs $H_1$, $H_2$ both with parts $Y$ and $Z$ such that $E(H)$ is the disjoint union of $E(H_1)$ and $E(H_2)$, every vertex in $Y$ has degree less than $d$ in $H_1$ and every vertex in $Z$ has degree less than $d$ in $H_2$.*

*Proof.* Since $H$ has minimum degree less than $d$, there is a vertex $u$ in $H$ which has degree less than $d$. If $u \in Y$, let every edge in $H$ of the form $uv$ belong to $H_1$, otherwise let every edge of the form $uv$ belong to $H_2$. Set $H' = H - u$.
Since $H'$ has minimum degree less than $d$, there is a vertex $u'$ in $H'$ which has degree less than $d$. If $u' \in Y$, let every edge in $H'$ of the form $u'v$ belong to $H_1$, otherwise let every edge of the form $u'v$ belong to $H_2$. Set $H'' = H' - u'$.
Continue this procedure until all edges are placed in $H_1$ or $H_2$. It is easy to see that these graphs are suitable. ∎

**Lemma 4.13.** *Let $H$ be a bipartite graph with parts $Y$ and $Z$ so that every vertex in $Y$ has degree at most $D_1$ and every vertex in $Z$ has degree at most $D_2$. Then the largest eigenvalue of $H$ is at most $\sqrt{D_1 D_2}$.*

*Proof.* Exercise. ∎

**Lemma 4.14.** *Let $A$ and $B$ be symmetric real matrices with largest eigenvalues $\lambda$ and $\mu$. Then the largest eigenvalue of $A + B$ is at most $\lambda + \mu$.*

*Proof.* Exercise. ∎

*Proof of Lemma 4.11.* Define graphs $H_1$ and $H_2$ as in Lemma 4.12. By Lemma 4.13, both $H_1$ and $H_2$ have largest eigenvalue at most $\sqrt{d\Delta(H)}$. Hence, by Lemma 4.14, the largest eigenvalue of $H$ is at most $2\sqrt{d\Delta(H)}$. ∎

**Lemma 4.15.** *Let $H$ be a bipartite graph with parts $Y$ and $Z$. Let $f : Y \to \mathbb{R}$ be a function and let $g(z) = \sum_{y \in N_H(z)} f(y)$ for every $z \in Z$. Suppose that $H$ does not contain a non-empty subgraph with minimum degree at least $d$. Then*

$$\sum_{y \in Y} f^2(y) \geq \frac{1}{4d\Delta(H)} \sum_{z \in Z} g^2(z).$$

*Proof.* Let $Y = \{y_1, y_2, \ldots, y_t\}$, $Z = \{z_1, z_2, \ldots, z_k\}$, $\vec{x}^T = (f(y_1), f(y_2), \ldots, f(y_t), 0, \ldots, 0)$, and matrix $A$ denote the adjacent matrix of $H$. Then we have $(A\vec{x})^T = (0, \ldots, 0, g(z_1), g(z_2), \ldots, g(z_k))$.

Assume that the maximum eigenvalue of $A$ is $\lambda$, which implies that the maximum eigenvalue of $A^2$ is $\lambda^2$. Then we have

$$\frac{\vec{x}^T A^T A \vec{x}}{\vec{x}^T \vec{x}} \leq \max \frac{\vec{y}^T A^T A \vec{y}}{\vec{y}^T \vec{y}} = \lambda^2.$$

.

From Lemma 4.11 we know that $\lambda^2 \leq 4d\Delta(H)$. Since $\vec{x}^T A^T A \vec{x} \leq \lambda^2 \vec{x}^T \vec{x}$, we have

$$\sum_{y \in Y} f^2(y) \geq \frac{1}{4d\Delta(H)} \sum_{z \in Z} g^2(z).$$

∎

The next lemma is one of our key lemma.

**Lemma 4.16.** *Let $k$ be a fixed positive integer and let $G$ be a properly edge-colored non-empty graph on $n$ vertices. Suppose that for some $2 \leq l \leq k$ we have*

$$hom(C_{2l}, G) \geq c_k \Delta(G) hom(C_{2l-2}, G).$$

*where $c_k = 2^{18} k^7$. Then $G$ contains a rainbow $C_{2k}$.*

*Proof.* Call a pair $(x_1, x_{l+1})$ of vertices *nice* if the number of rainbow injectively homomorphic copies of $C_{2l}$ of the form $x_1 x_2 \ldots x_{2l} x_1$ is greater than $\left(1 - \frac{1}{\binom{4k}{2}}\right) hom_{x_1, x_{l+1}}(C_{2l}, G) = \left(1 - \frac{1}{\binom{4k}{2}}\right) (hom_{x_1, x_{l+1}}(P_l, G))^2$.

**Claim**. If the pair $(x_1, x_l)$ is *nice*, then there are at least $4k$ pairwise vertex-disjoint path between $x_1$ and $x_{l+1}$ such that no color appears more than once on these paths.

*Proof.* We randomly select two paths $P^1$ and $P^2$ from the set of $hom_{x_1,x_{l+1}}(P_l, G)$ uniformly. Since one rainbow and injective homomorphic copies of $C_{2l}$ consists of two vertex-disjoint path between $x_1$ and $x_{l+1}$ such that no color appears more than once, then by the definition of nice pair

$$\begin{aligned}
&\mathbb{P}(P^1 \text{ and } P^2 \text{ either not vertex-disjoint or not rainbow})\\
&= 1 - \mathbb{P}(P^1 \text{ and } P^2 \text{ are vertex-disjoint and rainbow})\\
&< \frac{1}{\binom{4k}{2}}.
\end{aligned}$$

Then we have

$$\begin{aligned}
\mathbb{P}(\text{claim not happen}) &< \binom{4k}{2}\mathbb{P}(P^1 \text{ and } P^2 \text{ either not vertex-disjoint or not rainbow})\\
&< 1.
\end{aligned}$$

So there exist $4k$ pairwise vertex-disjoint paths between $x_1$ and $x_{l+1}$ such that no color appears more than once on these paths. ∎

Let # denote the number of non-rainbow or non-injective homomorphic copies of $C_{2l}$ in $G$. By Lemma 4.7 and Lemma 4.10, we know that

$$\# \leq 32l(l\Delta(G)hom(C_{2l}, G)hom(C_{2l-2}, G))^{1/2} \leq \frac{32l^{3/2}}{c_k^{1/2}} hom(C_{2l}, G).$$

And by definition of #, we know that

$$\# \geq \sum_{(x_1,x_{l+1}) \text{ not nice}} \frac{1}{\binom{4k}{2}} (hom_{x_1,x_{l+1}}(P_l, G))^2.$$

Since

$$\sum_{(x_1,x_{l+1})} (hom_{x_1,x_{l+1}}(P_l, G))^2 = hom(C_{2L}, G).$$

Then we have

$$\begin{aligned}
\sum_x \sum_{(x,z) nice} (hom_{x,x_z}(P_l, G))^2 &\geq (1 - \binom{4k}{2}\frac{32l^{3/2}}{c_k^{1/2}}) hom(C_{2l}, G)\\
&\geq \frac{1}{2} hom(C_{2l}, G) \geq \frac{c_k}{2}\Delta(G)hom(C_{2l-2}, G)\\
&= \frac{c_k}{2}\Delta(G)\sum_x hom_x(C_{2l-2}, G).
\end{aligned}$$

So there exist $x$ satisfied

$$\sum_{z \in V(G)} \sum_{(x,z) \text{ nice}} (hom_{x,z}(P_l, G))^2 \geq \frac{c_k}{2}\Delta(G)\sum_x hom_x(C_{2l-2}, G). \tag{4.1}$$

Let $Z = \{z \in V(G) : (x,z) \text{ is nice}\}$ and $Y = V(G)$. Let $H$ be the bipartite graph with part $Z$ and $Y$ incident by graph $G$ (i.e. $z\ y \in E(H)$ with color $r$ if and only if $z\ y \in E(G)$ with color $r$) .
**Case 1.** If $H$ does not contain a non-empty subgraph with minimum degree at least $4k$.

Let $f(y) = hom_{xy}(P_{l-1}, G)$ for any $y \in Y$ and

$$g(y) = \sum_{y \in N_H(z)} f(y) = \sum_{y \in N_H(z)} hom_{xy}(P_{l-1}, G) = \sum_{y \in N_G(z)} hom_{xy}(P_{l-1}, G) = hom_{xz}(P_l, G).$$

Then by Lemma 4.15, we know that

$$hom_x(C_{2l-2}, G) = \sum_{y \in Y} f^2(y) \geq \frac{1}{16k\Delta(H)} \sum_{z \in Z} g^2(z) \geq \frac{1}{16k\Delta(H)} \sum_{z \in Z} (hom_{xz}(P_l, G))^2,$$

which is contradiction to (4.1).
**Case 2.** If $H$ contains a non-empty subgraph$H_1$ with minimum degree at least $4k$.

If we have a vertex-disjoint rainbow path $x_1 x_2 \dots x_t$ in $H_1$ with length $t$ ($t < 2k$), since $\delta(H_1) \geq 4k > t$, there exist an adjacent edge $x_t x_{t+1}$ of $x_t$ with the different color to the path and $x_{t+1}$ is distinct to $x_i$, then we get a vertex-disjoint rainbow path $x_1 \dots x_t x_{t+1}$ in $H_1$ with length $t+1$. Thus we can have a vertex-disjoint rainbow path $P_1$ with start point$z_0$ and endpoint$z_1$ in $Z$ of length $2k - 2\ell$.

By definition of $Z$ and the **Claim**, there are more than $4k$ pairwise vertex-disjoint paths between $x$ and $z_0$ with length $\ell$ such that pairwise no color appears more than once. There are at most $2k - 2\ell$ paths have same color to $P_1$ and at most $2k - 2l$ paths have same vertex to $P_1$. So there exists a path $P_2$ is rainbow and vertex-disjoint to $P_1$.

And there are at most $2k - l$ paths have same color to $P_1 \cup P_2$ and at most $2k - l$ paths have same vertex to $P_1 \cup P_2$ So there exists a path $P_3$ is rainbow and vertex-disjoint to $P_1 \cup P_2$, and we get a rainbow $C_{2k} = P_1 \cup P_2 \cup P_3$.

∎

**Corollary 4.17.** *Let $k$ be a fixed positive integer and let $G$ be a properly edge-colored non-empty graph on $n$ vertices. Suppose that for some $2 \leq j \leq k$ we have*

$$hom(C_{2j}, G) = \omega(n\Delta(G)^j).$$

*Then, for $n$ sufficiently large, $G$ contains a rainbow $C_{2k}$.*

*Proof.* Choose $L = \omega(1)$ such that $hom(C_{2j}, G) \geq L^j n(\Delta(G))^j$. Let $\ell$ be the smallest integer satisfying $\hom(C_{2\ell}, G) \geq L^\ell n(\Delta(G))^\ell$.

If $\ell = 1$ then $\hom(C_2, G) = 2e(G) < Ln\Delta(G)$ is contradiction, so $2 \leq \ell \leq j \leq k$. Since $\ell$ is smallest, now we have $\hom(C_2, G) \geq L\Delta(G) hom(C_{2\ell-2}, G)$. By Theorem 4.16, $G$ contains a rainbow $C_{2k}$.

∎

**Definition 4.18.** *Graph $G$ is $K$-almost regular if $\Delta(G) \geq K\delta(G)$.*

**Lemma 4.19** (Jiang-Seiver)**.** *Let $\epsilon, c$ be positive reals, where $\epsilon < 1$ and $c \geq 1$. Let $n$ be a positive integer that is sufficiently large as a function of $\epsilon$. Let $G$ be a graph on $n$ vertices with $e(G) \geq cn^{1+\epsilon}$. Then $G$ contains a $K$-almost regular subgraph $G'$ on $m \geq n^{\frac{\epsilon-\epsilon^2}{2+2\epsilon}}$ vertices such that $e(G') \geq \frac{2c}{5} m^{1+\epsilon}$ and $K = 20 \cdot 2^{1+\frac{1}{\epsilon^2}}$.*

**Definition 4.20.** *The rainbow Turán number* $\mathrm{ex}^*(n, H)$ *is the maximum edges in a properly edge-colored graph on* $n$ *vertices which doesn't have rainbow* $H$ *as a subgraph.*

**Remark 4.21.** *Since a* $H$*-free graph must be a rainbow* $H$*-free graph, then we have* $\mathrm{ex}^*(n, H) \geq \mathrm{ex}(n, H)$.

**Theorem 4.22** (Keevash-Mubayi-Sudakov-Verstraëte)**.** *For all graph* $H$*, we have* $\mathrm{ex}^*(n, H) \leq ex(n, H) + o(n^2)$.

**Theorem 4.23** (Main Theorem)**.** $\forall k \geq 2$ $\mathrm{ex}^*(n, C_{2k}) = \Theta(n^{1+\frac{1}{k}})$.

*Proof.* The lower bound was obtained by Keevash-Mubayi-Sudakov-Verstraëte (Rainbow Turán Problems' TH1.3). We only prove the upper bound (i.e. $\mathrm{ex}^*(n, C_{2k}) \leq O(n^{1+\frac{1}{k}})$).

We assume that there exists $n$-vertex properly edge colored graph $G$ with more than $\omega(n^{1+\frac{1}{k}})$ edges that does not have rainbow $C_{2k}$ as a subgraph.

By Lemma 4.19 there exists a $K$-almost regular subgraph $H$ in $G$ on $m$ vertices with more than $\omega(\frac{2c}{5}m^{1+\frac{1}{k}})$ edges.

From $\omega(\frac{2c}{5}m^{1+\frac{1}{k}}) \leq e(H) \leq m\Delta(H) \leq mK\delta(H)$, we know $\delta(H) = \omega(m^{\frac{1}{k}})$. Since $C_{2k}$ satisfying Sidovenko-Conjecture, we have

$$\hom(C_{2k}, H) \geq \frac{(\hom(K_2, H))^{2k}}{m^{2k}} = \frac{(2e(G))^{2k}}{m^{2k}} \geq \delta^{2k} \geq (\frac{\delta^k}{mK^k})m(\Delta(H))^k = \omega(m(\Delta(H))^k).$$

which is contradiction to Corollary 4.17.

∎

# 5 Even-cycle-free Subgraphs of the Hypercube

The *n-dimensional hypercube*, $Q_n$, is the graph whose vertex set is the family $\{0,1\}^n$ and whose edge set is the set of pairs that differ in exactly one coordinate.

Here are some observations on $Q_n$ as follows:

(i) $Q_n$ is an $n$-regular graph with $2^n$ vertices and $n \cdot 2^{n-1}$ edges.

(ii) Each vertex can also be defined by some subset $S \subseteq [n]$ as follows: $\forall i$: $1 \leq i \leq n$,

$$\vec{v}_S(i) = \begin{cases} 1, & i \in S \\ 0, & i \notin S \end{cases} .$$

(iii) According to the definition above, we can label each vertex of $Q_n$ with all subsets of $[n]$ to get a layered structure of $Q_n$: serving the families of subsets $\varnothing$ , $\binom{[n]}{1}$, $\cdots$ , $\binom{[n]}{n-1}$, $[n]$ as layers respectively. For $S \in \binom{[n]}{i}$ and $T \in \binom{[n]}{j}$,

$$ST \in E(Q_n) \Rightarrow \ |i-j| = 1.$$

For graphs $Q$ and $P$, let $\text{ex}(Q,P)$ denote the *generalized Turán number*, i.e., the maximum number of edges in a $P$-free subgraph of $Q$. In particular, the *Turán number*, $\text{ex}(n,P) = \text{ex}(K_n,P)$.

Let $c_{2\ell}(n) = \frac{\text{ex}(Q_n,C_{2\ell})}{e(Q_n)} = \frac{\text{ex}(Q_n,C_{2\ell})}{n\cdot 2^{n-1}}$ and $c_{2\ell} = \lim\limits_{n\to+\infty} c_{2\ell}(n)$.

Note that $c_{2\ell}$ exists. (Homework. Hints: Only need to verify that $c_{2\ell}(n)$ is a non-increasing and bounded function of $n$.) The following conjecture of Erdös is still open.

**Conjecture 5.1** ( Erdös)**.** $c_4 = \frac{1}{2}$.

Erdös also asked if $\text{ex}(Q_n, C_{2\ell}) = o(n \cdot 2^n)$ for $\ell > 2$, i.e., whether $o(n\cdot 2^n)$ edges in a subgraph of $Q_n$ would imply the existence of a cycle $C_{2\ell}$ for $\ell > 2$.

Here are some known results as follows:

- (Thomason-Wagner) $c_4 \leq 0.6226$.
- (Brass-Harborth-Nienborg) $c_4(n) \geq \frac{1}{2}(1 + \frac{0.9}{\sqrt{n}})$ for $n \geq 9$.
- (Conder-Lu) For $C_6$, $\frac{1}{3} \leq c_6 < 0.3941$, due to Conder and Lu, respectively.
- (Chung) For $\ell \geq 2$, $c_{4\ell}(n) \leq cn^{-\frac{1}{2}+\frac{1}{2\ell}}$.

Next, we give the result to all $c_{4\ell+2}$ for $\ell \geq 3$ by Füredi and Özkahya.

**Theorem 5.2** (Füredi-Özkahya)**.** *For $\ell \geq 3$, then*

$$c_{4\ell+2}(n) = \begin{cases} O(n^{-\frac{1}{2\ell+1}}), & \ell \in \{3,5,7\}, \\ O(n^{-\frac{1}{16}+\frac{1}{16(\ell-1)}}), & \textit{otherwise.} \end{cases}$$

*i.e.,* $c_{4\ell+2} = 0$.

Combine the known results and 5.2, we see that:

$$\begin{cases} c_{2\ell} > 0, & \ell \in \{2,3\}, \\ c_{2\ell} = 0, & \ell \in \{4\} \cup \{6,7,8,...\}. \end{cases}$$

It is still open to consider if $c_{10} = 0$ or not. We will give two proofs about $c_{2\ell}$ for $\ell \in \{4\} \cup \{6,7,8,...\}$: One is proved by Füredi and Özkahya, another is by Conlon.

### 5.1 The First Proof of Theorem 5.2 by Füredi -Özkahya

For $k \geq 3$, let $G$ be any $C_{4k+2}$-free subgraph of $Q_n$. Fix two integers $a, b \geq 2$ such that $4a + 4b = 4k+4$. Then any cycle of length $4a$ cannot intersect a cycle of length $4b$ at a single edge, otherwise their union contains a $C_{4k+2}$.

Let $N(G,P)$ denote the number of subgraphs of $G$ that are isomorphic to $P$. For an edge of $Q_n$: $uv \in E(Q_n)$, we define the *direction* of $uv$, denoted by $d(uv)$, to be the unique coordinate from $[n]$ where the 0-1 vectors $u$ and $v$ differ. Similarly, for a subgraph $F$ of $Q_n$, let $D(F) := \{d(e) : e \in E(F)\}$.

**Lemma 5.3.** *Let $C'$ and $C''$ be cycles of length $4a$ and $4b$ of $G$, respectively, whose intersection contains an edge. Then*

$$\left|D(C') \cap D(C'')\right| \geq 2.$$

*Proof.* Let $v_1$, $v_2$ be the endpoints of the common edge in the intersection of $C'$ and $C''$. By the previous observation, there must be another vertex $v_3$ common in $C'$ and $C''$. Because $v_3$ has at least two coordinates that differs from either $v_1$ or $v_2$, these two coordinates are also contained in the intersection of $D(C')$ and $D(C'')$.

∎

Observe that, for any cycle C of length $C_{2\ell}$ in $Q_n$,

$$|D(C_{2\ell})| \leq \ell.$$

Because the direction of each edge in $C_{2\ell}$ appears an even number of times on $E(C_{2\ell})$. Based on this observation, we can get the result directly as follows:

$$N(G, C_{4a}) \leq N(Q_n, C_{4a}) \leq 2^n \cdot O(n^{2a}).$$

Homework : Give a proof to explain the reason why the last equality holds.

In the following, we obtain a better bound of $N(G, C_{4a})$.

**Lemma 5.4.**

$$N(G, C_{4a}) = e(G) \cdot O(n^{2a-2}) + O(2^n n^{2a-\frac{1}{2}+\frac{1}{2b}}),$$

*where $G$, $a, b \geq 2$ are defined as before.*

*Proof.* Let $\mathcal{C}$ denote the family of all cycles of length $4a$ in $G$ and let $\mathcal{E}_0$ be the set of edges contained in the cycles in $\mathcal{C}$. We partition $\mathcal{E}_0 = \mathcal{E}_1 \cup \mathcal{E}_2$, where $\mathcal{E}_1$ is the collection of edges that are contained in the intersection of a copy of $C_{4a}$ and a copy of $C_{4b}$ in $G$, and $\mathcal{E}_2 := \mathcal{E}_0 \setminus \mathcal{E}_1$. Next, we count the cycles of length $4a$ in G over the edges in $\mathcal{E}_0$.

By Lemma 5.3, every edge $e \in \mathcal{E}_1$ is contained in $O(n^{2a-2})$ members of $\mathcal{C}$.(As the definition of $\mathcal{E}_1$, for any cycle $C_{4a}$ contains $e$ fixed, there exist a cycle $C_{4b}$ contains $e$. For $e \in \mathcal{E}_1$, there are at most $\binom{2b}{2}$ choices of the directions arranged to $e$, then according to Lemma5.3 about the fixed cycle $C_{4a}$, there are at most $\binom{n}{2a-2}$ choices of the directions to choose for the directions of all edges of $C_{4a}$ except $e$. Thus the edge $e$ is contained in at most $\binom{2b}{2}\binom{n}{2a-2} = O(n^{2a-2})$ members in $\mathcal{C}$.)

The subgraph induced by the edges in $\mathcal{E}_2$ does not contain a copy of $C_{4b}$, implying that

$$|\mathcal{E}_2| \leq \mathrm{ex}(Q_n, C_{4b}) \leq O(2^n n^{\frac{1}{2}+\frac{1}{2b}}),$$

where the last inquality is from Chung's Theorem from the known results above (slightly expand the magnitude of $n$ for some reasons).

Using these bounds, we obtain

$$\begin{aligned} N(G, C_{4a}) &= \frac{1}{4a} \sum_{e \in \mathcal{E}_0} (\textit{the number of cycles } C_{4a} \textit{ containing } e) \\ &= \frac{1}{4a} \left( \sum_{e \in \mathcal{E}_1} O(n^{2a-2}) + \sum_{e \in \mathcal{E}_2} O(n^{2a-1}) \right) \\ &\leq e(G) \cdot O(n^{2a-2}) + O(2^n n^{2a - \frac{1}{2} + \frac{1}{2b}}). \end{aligned}$$

∎

**Lemma 5.5.** *Let* $\bar{d} = \dfrac{2e(G)}{2^n}$*. Then,* $N(G, C_{4a}) \geq c 2^n \dfrac{\bar{d}^{4a}}{n^{2a}} - O(2^n n^a)$ *for some* $c > 0$*.*

**Theorem 5.6** (Known many years ago)**.** $\forall a \geq 2, \exists c, c' > 0$*, such that for any* $n$*-vertex graph* $G$ *with* $e \geq cn^{1+\frac{1}{a}}$ *edges,* $N(G, C_{2a}) \geq c' \left(\dfrac{e}{n}\right)^{2a}$*.*

**Remark 5.7.** *This says that the supersaturation conjecture of Erdös-Simonouts for even cycles is true.*

*Proof of Lemma 5.5.* For a graph $G \subseteq Q_n$, for any $x \in V(Q_n)$, we define an auxiliary $H_x = H_x(G)$ as follows: $V(H_x)$ consists of all neighbors of $x$ in $Q_n$ and $E(H_x)$ consists of pairs $yz$ such that there exists a vertex $w = w(y, z)$ satisfying that $yw, zw \in E(G)$. Note that as vectors over $\mathbb{F}_2$, we have $w = y + z - x$. So $yz \in E(H_x)$ if and only if $wz, wy \in E(G)$. We have

$$\sum_{x \in V(Q_n)} e(H_x) = \sum_{w \in V(G)} \binom{d_G(w)}{2} \geq 2^n \binom{\bar{d}}{2}.$$

**Fact 1.** For each $C_l$ in $H_x$, say $y_1 y_2 \cdots y_l y_1$, there exists a $C_{2l}$ in $G$, namely $y_1, w(y_1, y_2), y_2, w(y_2, y_3), \cdots, y_l, w(y_l, y_1)$.

**Fact 2.** All cycles in different $H_x$ and $H_y$ are distinct.

Thus,

$$N(G, C_{4a}) \geq \sum_{x \in V(Q_n)} N(H_x, C_{2a}).$$

By Theorem 5.6, we have

$$N(H_x, C_{2a}) \geq c' \left(\frac{e(H_x)}{n}\right)^{2a} - c' \left(\frac{cn^{1+\frac{1}{a}}}{n}\right)^{2a}.$$

Combining above two inequalities, we have

$$N(G, C_{4a}) \geq c' \sum_{x \in V(Q_n)} \frac{(e(H_x))^{2a}}{n^{2a}} - O(2^n n^2).$$

Let $\bar{h} = \dfrac{\sum e(H_x)}{2^n}$, so $\bar{h} \geq \binom{\bar{d}}{2}$. By Jensen's inequality, we have

$$N(G, C_{4a}) \geq \frac{c' 2^n}{n^{2a}} (\bar{h})^{2a} - O(2^n n^2) \geq c'' 2^n \frac{\bar{d}^{4a}}{n^{2a}} - O(2^n n^2).$$

∎

*Proof of Theorem 5.2.* Let $G \subseteq Q_n$ be $C_{4k+2}$-free. Fix $a, b \geq 2$ as before. By Lemma 5.4 and 5.5, we have

$$c 2^n \frac{(\bar{d})^{4a}}{n^{2a}} \leq O(2^n n^2) + \bar{d} 2^n O(n^{2a-2}) + O(2^n n^{2a - \frac{1}{2} + \frac{1}{2b}}),$$

which implies that $\bar{d} = \max\{O(n^{1-\frac{1}{4a-1}}), O(n^{1-\frac{1}{4a}(\frac{1}{2}-\frac{1}{2b})})\}$. Where $a, b$ are any integers with $a, b \geq 2$ and $a + b = k + 1$. Take $a = 2$ and $b = k - 1 \geq 2$. Then we get $\bar{d} = O(n^{1-\frac{1}{16}+\frac{1}{16(k-1)}})$. This implies that $C_{2k+1} = 0$ for any $k \geq 3$. ∎

**Observations.**

1. For $k \in \{3, 5, 7\}$, take $a = b = \frac{k+1}{2}$. This will change the argument in the proof of Lemma 1.1, the number of copies of $C_{4a}$ contains some edges in $E_2$ is at most $\dfrac{e(G)}{4a}$. $\Rightarrow \bar{d} = O(n^{1-\frac{1}{2k-1}})$.
2. Let $\Theta_{u,v,w}$ be a Theta-graph consisting of 3 internally disjoint paths of lengths $u, v, w$ between 2 vertices. Our proof in fact tells that $\mathrm{ex}(Q_n, \Theta_{4a-1,1,4b-1}) = o(e(Q_n))$.
3. The proof of $C_{2k} = 0$ is from the proof of Lemma 5.5.

   $C_{2k} = 0$ for $k \geq 2$.

## 5.2 The Second Proof by D. Conlon

There is a one-to-one correspondence between $V(Q_n)$ and subsets of $[n]$. Any edge in $Q_n$ can be written uniquely by a sequence like $[00101 * 100 \cdots 1]$. The missing bit * is called the flip-bit.

**Definition 5.8.** *We say that a subgraph $H$ of the hypercube $Q_n$ has a $k$-partite representation, if there exists some $\ell$ such that (a) $H \subseteq Q_\ell$; (b) every edge $e = a_1 a_2 \cdots a_\ell$ in $H$ has exactly $k$ non-zero bits (i.e. $k-1$ ones and a flip-bit); (c) there exists a function $\sigma : [\ell] \to [k]$, such that for $e \in E(H)$, the image $\{\sigma(i_1), \cdots, \sigma(i_k)\}$ of the set of non-zero bits $\{a_{i_1}, \cdots, a_{i_k}\}$ of under $\sigma$ is $[k]$.*

**Definition 5.9.** *For a subgraph $H \subseteq Q_n$, let $\mathscr{H}$ be the $k$-uniform hypergraph on $[\ell]$ with edge set $\{\tau(e) : e \in H\}$, where the function $\tau$ is defined by mapping the set of non-zero bits $\{a_{i_1}, a_{i_2}, \cdots, a_{i_k}\}$ of $e$ to the subset $\{i_1, i_2, \cdots, i_k\}$ of $[\ell]$. We refer $\mathscr{H}$ as a ($k$-partite) representation of $\mathscr{H}$.*

**Example 5.10.** *$C_8$ has a 2-partite representation.*

$\ell = 4, C_8 \subseteq Q_4 . v_1 = [1,1,0,0], v_2 = [0,1,0,0], v_3 = [0,1,1,0], v_4 = [0,0,1,0], v_5 = [0,0,1,1], v_6 = [0,0,0,1], v_7 = [1,0,0,1], v_8 = [1,0,0,0]$.

**Homework:** verify that any $C_{4k}$ for $k \geq 4$ has a 2-partite representation.

**Example 5.11.** *$C_{14}$ has a 3-partite representation.*

$\ell = 7. V(C_{14}) = \{1110000, 0110000, 0111000, 0011000, 0011100, 0001100, 0001110, 0000110, 0000111, 0000011, 1000011, 1000001, 1100001, 1100000\}$. $E(C_{14}) = \{*110000, 011*000, 01*1000, 0101*00, 010*100, *100100, 1*00100, 10001*0, 1000*10, 100001*, 10000*1, 1*00001, 110000*, 11*0000\}$. *For this $C_{14}$ in $Q_7$, there is no $\sigma : [7] \to [7]$ satisfying the (c).*

**Homework:** Any $C_{4k+2} (k \geq 3)$ has a $3-partite$ representation.

$C_{4k} (k \geq 2) \to$ 2-partite, $C_{4k+2} (k \geq 3) \to$ 3-partite, $C_{10}$ has no $k$-partite for all $k$.

**Remark 5.12.** *$C_4, C_6, C_{10}$ do not admit $k$-partite representation for any $k$.*

This is the second proof of the Chung-Fredi-zkahya Theorem. We state the Conlon Theorem as follows.

**Theorem 5.13** (Conlon)**.** *Let $H$ be a fixed subgraph of a hypercube. If there exists some integer $k \geq 2$ such that $H$ admits a $k$-partite representation, then*

$$\text{ex}(Q_n, H) = o(e(Q_n)).$$

First we need the following lemma.

**Lemma 5.14.** *The number of subsets of $[n]$ containing fewer than $\frac{n}{4}$ or more than $\frac{3n}{4}$ is at most $(1.9)^n n$.*

*Proof.* Then number of counted subsets is at most

$$2 \sum_{j=0}^{\frac{n}{4}} \binom{n}{j} \leq n \binom{n}{\frac{n}{4}} \leq (4e)^{\frac{n}{4}} n \leq (1.9)^n n,$$

the second inequality is from the estimate $j! \geq \left(\frac{j}{e}\right)^j$. ∎

We also need the following classical result of Erdös, regarding the extremal number of complete $k$-partite $k$-uniform hypergraphs

**Lemma 5.15.** *Let $K^{(k)}_{(s_1, \cdots, s_k)}$ be the complete $k$-partite $k$-uniform hypergraph with parts of sizes $s_1, s_2, \cdots, s_k$. Then $\text{ex}(n, K^{(k)}_{(s_1, \cdots, s_k)}) = O(n^{k-\delta})$ where $\delta = (s_1 \cdots s_{k-1})^{-1}$.*

**Theorem 5.16.** *Let $H$ be a subgraph of the cube with $k$-partite representation $\mathcal{H}$. Suppose $\text{ex}(n, \mathcal{H}) \leq \alpha n^k$, then $\text{ex}(Q_n, H) \leq O(\alpha^{\frac{1}{k}} 2^n n)$.*

*Proof.* Let $G$ be a subgraph of $Q_n$ with edge density $\epsilon := \frac{e(G)}{e(Q_n)} = 16k\alpha^{\frac{1}{k}}$. Our goal is to show that $G$ contains a copy of $H$.

By Lemma 5.14, the set $A$ of vertices containing fewer than $\frac{n}{4}$ or more than $\frac{3n}{4}$ ones satisfy $|A| \leq (1.9)^n n$. Since $Q_n$ is $n$-regular, the total number of edges incident to $A$ is at most $(1.9)^n n^2$, which is very fewer, so the edge density of $G - A$ is at least $\frac{\epsilon}{2}$.

This implies that there exists some $\frac{n}{4} \leq j \leq \frac{3n}{4}$ such that the edge density in $G$ between level $j$ and $j+1$ is at least $\frac{\epsilon}{2}$. So there are at least $\frac{\epsilon}{2}(n-j)\binom{n}{j} \geq \frac{\epsilon}{2} j \binom{n}{j+1}$ edges in $G$ between level $j$ and $j+1$. Let $G_j$ be the subgraph of $G$ induced by the vertices in level $j$ and $j+1$.

Any edge between level $j$ and $j+1$ can be viewed as a collection of $j$ ones and a flip-bit. For any $J \in \binom{[n]}{j+1}$, let $D(J)$ be the subset of $J$ consisting integers $a$ such that there exists an edge of $G$ whose flip-bit is $a^{th}$ position. Let $d(J) = |D(J)|$ (i.e. the degree of $J$). So the average degree of vertices in the $(j+1)$-level in the subgraph $G_j$ is at least $\frac{\epsilon}{2}j$ (i.e. $\overline{d(J)} \geq \frac{\epsilon}{2}j$).

Now we have the

$$\sum_{J \in \binom{[n]}{j+1}} \binom{d(J)}{k} \geq \binom{n}{j+1}\binom{\overline{d(J)}}{k} \geq \binom{n}{j+1}\binom{\frac{\epsilon}{2}j}{k}$$

This tells us that there are at least $\binom{n}{j+1}\binom{\frac{\epsilon}{2}j}{k}$ pairs of $(I, J)$ for which $|J| = j+1, |I| = k$ and $I \subseteq D(J)$.

By averaging, there exists some set $S$ of size $j+1-k$ for which at least $\binom{n}{j+1}\binom{\frac{\epsilon}{2}j}{k}/\binom{n}{j+1-k} = \binom{n-j-1+k}{k}\binom{\frac{\epsilon}{2}j}{k}/\binom{j+1}{k}$ pairs of $(I, J)$ satisfy $J \setminus I = S$. Note that for fixed $S$, the pair of $(I, J)$ is uniquely determined by the choice of $I$. Let $\mathcal{I}$ be the $k$-uniform hypergraph whose edges are taken from these pairs.

Since $\frac{4k}{\epsilon} \leq \frac{n}{4} \leq j \leq \frac{3n}{4}$, by $\alpha n^k \geq 1$, we see that the number of edges in $\mathcal{I}$ is at least $\binom{n-j-1+k}{k}\binom{\frac{\epsilon}{2}j}{k}/\binom{j+1}{k} \geq \left(\frac{\epsilon}{4}\right)^k \binom{n-j-1+k}{k} \geq \left(\frac{\epsilon}{16}\right)^k \frac{n^k}{k!} \geq \alpha n^k$. Since $\text{ex}(n, \mathcal{H}) \leq \alpha n^k$, the hypergraph $\mathcal{I}$ contains a copy of $\mathcal{H}$. By the definition of $\mathcal{H}$, there is a copy of $H$ in $G$.

Note the $V(\mathcal{H}) = [\ell]$. So we have a mapping $g : [\ell] \to [n]$ describes the embedding of $\mathcal{H}$ in $\mathcal{I}$. We define a mapping $f : V(Q_\ell) \to V(Q_n)$ by mapping $[a_1 a_2 \cdots a_\ell]$, with non-zero bits $a_{i_1} \cdots a_{i_r}$, to $[b_1 b_2 \cdots b_n]$ such that $b_i = 1$ if and only if $i \in S \cup \{g(i_1), \cdots, g(i_r)\}$. Note that this is a graph isomorphism between $Q_\ell$ and $f(Q_\ell)$.

**Claim**: For any $e = uv \in E(H) \subseteq E(Q_l)$, the edge $f(u)f(v) \in E(G)$.

*Proof.* Suppose the non-zero bits of $e$ are $a_{i_1}, a_{i_2}, \cdots, a_{i_k}$ and the flip-bit is $a_{i_t}$ $(1 \leq t \leq k)$. Let $J = S \cup \{g(i_1), \cdots, g(i_k)\}$. By construction of $D(J)$, the edge formed by replacing $b_{g(i_t)}$ in the vector $f(u)$ with a flip-bit is in $G$. But this edge is just the edge between $f(u)$ and $f(v)$ for actually any $i_t$. This proves the claim and completes the proof. ∎

*Proof the Conlon Theorem.* By Lemma 5.15 and Theorem5.16, we see that any $H$ with a $k$-partite representation satisfies $\text{ex}(Q_n, H) = o(e(Q_n))$. In particular $C_{2\ell}$ for $\ell \in \{4, 6, 7, \cdots\}$ satisfies this. ∎

# 6 Spectral Graph Theorem

**Definition 6.1.** *Let $M$ be a symmetric real matrix, the Rayleigh quotient of a vector $\vec{x}$ with respect to $M$ is defined by $\frac{\vec{x}^t M \vec{x}}{\vec{x}^t \vec{x}}$.*

**Theorem 6.2.** *Let $\vec{x}$ be a non-zero vector that maximizes the Rayleigh quotient with respect to $M$. Then, $\vec{x}$ is an eigenvector of $M$ corresponding to the largest eigenvalue of $M$, which is $\max_{\vec{x} \neq \vec{0}} \frac{\vec{x}^t M \vec{x}}{\vec{x}^t \vec{x}} = \lambda_{max}$.*

*Proof.* First, we realize that it is sufficient to consider unit vectors $\vec{x}$ as Rayleigh quotient is homogeneous, and the set of unit vectors is a closed and compact set, implying that $\lambda_{max} = \max_{\vec{x} \neq \vec{0}} \frac{\vec{x}^t M \vec{x}}{\vec{x}^t \vec{x}}$ exists. Let $\vec{x}$ be such a vector achieving this maximum. Note that

$$\nabla \vec{x}^t \vec{x} = 2\vec{x},$$

and

$$\nabla \vec{x}^t M \vec{x} = 2M\vec{x}.$$

$$\nabla(x_1^2, x_2^2, ..., x_n^2) = \begin{pmatrix} 2x_1 \\ 2x_2 \\ \vdots \\ 2x_n \end{pmatrix}.$$

Then

$$\nabla \frac{\vec{x}^t M \vec{x}}{\vec{x}^t \vec{x}} = \frac{(\nabla \vec{x}^t M \vec{x})\vec{x}^t \vec{x} - (\vec{x}^t M \vec{x})\nabla \vec{x}^t \vec{x}}{(\vec{x}^t \vec{x})^2}$$
$$= \frac{(\vec{x}^t \vec{x})(2M\vec{x}) - (\vec{x}^t M \vec{x})(2\vec{x})}{(\vec{x}^t \vec{x})^2}.$$

Since the gradient of a function at its maximum should be the zero vector, we get

$$(\vec{x}^t \vec{x})(2M\vec{x}) = (\vec{x}^t M \vec{x})(2\vec{x}).$$

Then

$$M\vec{x} = (\frac{\vec{x}^t M \vec{x}}{\vec{x}^t \vec{x}})\vec{x} = \lambda_{max}\vec{x}.$$

It is easy to see that $\lambda_{max}$ is at least all other eigenvalue of $M$.

∎

## 6.1 Courant-Fischer Theorem

**Theorem 6.3** (Courant-Fischer Theorem)**.** *Let $M$ be a symmetric real matrix with eigenvalues $\lambda_1 \le \lambda_2 \le ... \le \lambda_n$. Then*

$$\lambda_k = \min_{S \subset \mathcal{R}^n, dim(S)=k} \max_{\vec{x} \in S} \frac{\vec{x}^t M \vec{x}}{\vec{x}^t \vec{x}} = \min_{T \subset \mathcal{R}^n, dim(T)=n-k+1} \max_{\vec{x} \in T} \frac{\vec{x}^t M \vec{x}}{\vec{x}^t \vec{x}}.$$

*Remark: when $k = n$, it is just the previous result.*

*Proof.* Let $\vec{\varphi_1}, \vec{\varphi_2}, ..., \vec{\varphi_n}$ be an orthonormal set of eigenvectors of $M$ corresponding to $\lambda_1, \lambda_2, ..., \lambda_n$ respectively. We will only prove the first equality for $\lambda_k$ (the other is similar).

(i) Let $S_k$ be the span of $\vec{\varphi_1}, \vec{\varphi_2}, ..., \vec{\varphi_n}$. Then any $\vec{x} \in S_k$ can be expanded as $\vec{x} = \sum_{i=1}^{k} c_i \vec{\varphi_i}$. Then

$$\frac{\vec{x}^t M \vec{x}}{\vec{x}^t \vec{x}} = \frac{\sum_{i=1}^{k} c_i^2 \vec{\varphi_i}}{\sum_{i=1}^{k} c_i^2} \le \lambda_k.$$

This shows that

$$\lambda_k \ge \max_{\vec{x} \in S_k} \frac{\vec{x}^t M \vec{x}}{\vec{x}^t \vec{x}} \ge \min_{S \subset \mathcal{R}^n, dim(S)=k} \max_{\vec{x} \in S} \frac{\vec{x}^t M \vec{x}}{\vec{x}^t \vec{x}}.$$

(ii) Consider any subspace $S$ with $k$ dimension. We need to show $\lambda_k \leq \max\limits_{\vec{x}\in S} \frac{\vec{x}^t M\vec{x}}{\vec{x}^t\vec{x}}$. Let $T_k$ be the span of $\vec{\varphi_k}, \vec{\varphi_{k+1}}, ..., \vec{\varphi_n}$. So dim $(T_k) = n - k + 1$. Thus, $dim(S \cap T_k) \geq 1$, and

$$\max_{\vec{x}\in S} \frac{\vec{x}^t M\vec{x}}{\vec{x}^t\vec{x}} \geq \max_{\vec{x}\in S\cap T_k} \frac{\vec{x}^t M\vec{x}}{\vec{x}^t\vec{x}}.$$

Any $\vec{x} \in S \cap T_k$ can be expressed as $\vec{x} = \sum\limits_{i=k}^{n} c_i\vec{\varphi_i}$ . So

$$\frac{\vec{x}^t M\vec{x}}{\vec{x}^t\vec{x}} = \frac{\sum\limits_{i=k}^{n} c_i^2\vec{\varphi_i}}{\sum\limits_{i=k}^{n} c_i^2} \geq \lambda_k.$$

We are done! ∎

**Exercise 1.** *Prove the* $2^n d$ *equality.*

**Theorem 6.4.**

$$\lambda_i = \min_{\vec{x}\perp span(\vec{\varphi_1},\vec{\varphi_2},...,\vec{\varphi_n})} \frac{\vec{x}^t M\vec{x}}{\vec{x}^t\vec{x}},$$

*and* $\vec{\varphi_i} \in arg(\lambda_i)$.

*Proof.* Exercise. ∎

**Definition 6.5.** *Let* $G$ *be an* $n$*-vertex graph. Let* $A_G$ *be its adjacency matrix* $L_G$ *be its Laplacian matrix. We often assume that* $0 = \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n$ *are the eigenvalues of* $L_G$*, and* $\mu_1 \geq \mu_2 \geq ... \geq \mu_n$ *are the eigenvalues of* $A_G$.

**Proposition 6.6.** $\overline{d}(G) \leq \mu_1 \leq \Delta(G)$ *, where* $\overline{d}(G)$ *denotes the average degree of* $G$.

*Proof.* Recall Theorem 6.3 that

$$\mu_1 = \max_{\vec{x}\neq\vec{0}} \frac{\vec{x}^t M\vec{x}}{\vec{x}^t\vec{x}} \geq \frac{\vec{1}^t M\vec{1}}{\vec{1}^t\vec{1}} = \frac{2e(G)}{n} = \overline{d}(G).$$

Let $\vec{v}$ be the eigenvector of $\mu_1$ and suppose its largest entry is its first entry say $a$. (verify $a > 0$) Then

$$A \cdot \vec{v} = \mu_1 \cdot \vec{v} \Leftrightarrow \begin{pmatrix} * * * * * \\ \\ \\ \\ \end{pmatrix} \begin{pmatrix} a \\ \cdot \\ \cdot \\ \cdot \end{pmatrix} = \mu_1 \cdot \begin{pmatrix} a \\ \cdot \\ \cdot \\ \cdot \end{pmatrix}$$

Let $\vec{r_1}$ be the first row of $A$. Then

$$\mu_1 a = \vec{r_1}\vec{v} \leq \Delta(G)a,$$

which implies that $\mu_1 \leq \Delta(G)$. ∎

**Proposition 6.7.** *If* $G$ *is connected, then* $\mu_1 = \Delta(G)$ *if and only if* $G$ *is regular.*

*Proof.* Exercise. ∎

**Proposition 6.8.** *If $G$ is regular and not connected, then $\mu_1 = \mu_2$.*

*Proof.* Exercise. ∎

**Proposition 6.9.** *If $G$ is connected, then $\mu_1 > \mu_2$.*

*Proof.*

**Claim 1.** *Let $\vec{v_1}$ be any eigenvector of $\mu_1$. Then $\vec{v_1}$ is a non-negative vector.*

*Proof.* $\max \frac{\vec{x}^t M \vec{x}}{\vec{x}^t \vec{x}}$ is maximized by a non negative $\vec{x}$. ∎

**Claim 2.** *If $G$ is connected, then $\vec{v_1} > 0$.*

*Proof.* In fact, we can show that if $G$ is connected and $\vec{v} \geq (\vec{v} \neq \vec{0})$ is an eigenvector, then $\vec{v} > 0$. To see this, consider $A^k_{i,j}$, which is the number of walks in $G$ of length $k$ between $i$ and $j$. Since $G$ is connected, there exists some $k$ such that $A^k_{i,j} > 0$ for all $i, j \in V(G)$. (Why?) Consider $A^k \vec{v} = \lambda^k \vec{v}$, where $\lambda$ is the eigenvalue of $\vec{v}$.

$$\vec{r_j} \rightarrow \begin{pmatrix} \\ * * * * * \\ \\ \end{pmatrix} \cdot \vec{v} = \lambda^k \cdot \begin{pmatrix} \\ * \\ \\ \end{pmatrix} \rightarrow j^{th}$$

This proves Claim 2. ∎

Let $\vec{v_2}$ be the eigenvector of $\mu_2$. Then $\vec{v_1} \cdot \vec{v_2} = 0$. This says that $\vec{v_2}$ has a negative entry, and thus $\vec{v_2}$ can not maximize the Rayleigh quoitent $\max\limits_{\vec{x}} \frac{\vec{x}^t M \vec{x}}{\vec{x}^t \vec{x}}$. So $\mu_2 \neq \mu 1 \Rightarrow \mu_2 < \mu_1$ .

∎

## 6.2 Cheeger's Inequality

**Definition 6.10.** *Let $S \subset V(G)$. The boundary of $S$, denoted by $\partial S$, is $\partial S = \{ab \in E(G) : a \in S, b \notin S\}$. The isoperimetric ratio of $S$ is defined to be*

$$\theta(S) = \frac{|\partial S|}{|S|}$$

*.The isoperimetric number of $G$ is defined to be*

$$\theta_G = \min_{|S| \leq \frac{|V(G)|}{2}} \theta(S).$$

**Theorem 6.11** (Cheeger's inequality)**.** *For any $S \subset V(G)$,$\theta(S) \geq \lambda_2(1 - s)$, where $s = \frac{|S|}{|V(G)|}$. In particular, $\theta \geq \frac{\lambda_2}{2}$. ($\sqrt{2\lambda_2} \geq \theta_G \geq \frac{\lambda_2}{2}$)*

*Proof.* In $L_G$, we know $\lambda_1 = 0$ and $\vec{1}$ is the eigenvector of $\lambda_1$ in $L_G$. So

$$\lambda_2 = \min_{\vec{x}:\vec{x}\cdot\vec{1}=0} \frac{\vec{x}^t L_G \vec{x}}{\vec{x}^t \vec{x}}$$

This tells that for every non-zero $\vec{x}$which is orthogonal to $\vec{1}$satisfies that $\vec{x}^t L_G \vec{x} \geq \lambda_2 \vec{x}^t \vec{x}$. So we need a vector related to $S$. Let the characteristic vector of $S$ be

$$\vec{\chi_s}(a) = \begin{cases} 1, & \text{if } a \in S \\ 0, & \text{otherwise} \end{cases}.$$

Take $\vec{x} = \vec{x_s} - s\vec{1}$ , where $s = \frac{|S|}{|V(G)|}$. Then we have $\vec{x} \cdot \vec{1} = 0$.

$$\vec{x}^t L_G \vec{x} = \sum_{i,j \in E(G)} (x(\vec{i}) - x(\vec{j}))^2 = \sum_{i,j \in E(G)} (x_s(\vec{i}) - x_s(\vec{j}))^2 = |\partial S|$$

$$\vec{x}^t \vec{x} = |S|(1-s)^2 + (|V| - |S|) \cdot s^2 = |S| \cdot (1-s)$$

Combining, we get

$$\lambda_2 \leq \frac{\vec{x}^t L_G \vec{x}}{\vec{x}^t \vec{x}} = \frac{|\partial S|}{|S|(1-s)} = \frac{\theta(s)}{1-s}.$$

∎

**Proposition 6.12.** *Let $S \subset V$ be a subset of size $S|V|$.Then*

$$\|\vec{x_s} - s\vec{1}\|^2 = s(1-s)|V|.$$

## 6.3 Graphic Inequalities

$$\theta_G = \min_{|S| \leq \frac{|N(G)|}{2}} \frac{|\partial S|}{|S|}.$$

We have shown that $\theta_G \geq \lambda_2/2$.
Given a weighted graph $G = (V, E, w)$, where $w : E \to \mathbf{R}^+$, it's Laplacian matrix is designed to capture the Laplacian quadratic form :

$$\vec{x}^T L_G \vec{x} = \sum_{ij \in E(G)} w_{ij} (\vec{x}(i) - \vec{x}(j))^2.$$

A symmetric matrix $A$ is positive semidefinite demoted by $A \succeq 0$, if all it's eigenvalues $u_1, u_2, \ldots, u_n$ are non-negative. That is, $\vec{v}^T A \vec{v} \geq 0$ for any $\vec{v} \in \mathbf{R}^n$.

We write $A \succeq B$, if $A - B \succeq 0$. That is, $\forall \vec{v} \in \mathbf{R}^n, \vec{v}^T A \vec{v} \geq \vec{v}^T B \vec{v}$.

Given two graphs $G$ and $H$ of the same number of vertices. we say $G \succeq H$, if $L_G \succeq L_H$.

For a positive constant $c$, we write $G \succeq cH$ if and only if $\forall \vec{v} \in \mathbf{R}^n$ , $\vec{v}^T G \vec{v} \geq c\vec{v}^T H \vec{v}$.

**Theorem 6.13.** *If $G$ and $H$ are graphs with $G \succeq cH$ for a positive constant $c$ , then for every $k$ , $\lambda_k(G) \geq c\lambda_k(H)$.*

*Proof.* The Courant-Fischer Theorem tells us that

$$\lambda_k(G) = \min_{S \subseteq \mathbf{R}^n, dim(S)=k} \max_{\vec{x} \in S} \frac{\vec{x}^T L_G \vec{x}}{\vec{x}^T \vec{x}} \geq c \min_{S \subseteq \mathbf{R}^n, dim(S)=k} \max_{\vec{x} \in S} \frac{\vec{x}^T L_H \vec{x}}{\vec{x}^T \vec{x}} = c\lambda_k(H).$$

∎

Let $G$ be a (weighted) graph and let $H$ be obtained by either adding an edge to $G$ or increasing the weight of an edge in $G$. Then, for all $k$, $\lambda_k(H) \geq \lambda_k(G)$.

**Theorem 6.14.** *Let $P_n$ be the path from vertex 1 to vertex $n$, and let $G_{i,j}$ be the graph with just the edge $ij$. Then*

$$(n-1)P_n \succeq G_{1,n}.$$

*Proof.* We need to show that for any $\vec{x} \in \mathbf{R}^n$,

$$\vec{x}^T L_{P_n} \vec{x} \geq \vec{x}^T L_{G1,n} \vec{x}.$$

That is

$$(n-1)\sum_{i=1}^{n-1} (\vec{x}(i+1) - \vec{x}(i))^2 \geq (\vec{x}(n) - \vec{x}(1))^2,$$

this can be obtained by Cauchy-Schwarz inequality. ∎

**Theorem 6.15.**

$$\frac{6}{(n+1)(n-1)} \leq \lambda_2(P_n) \leq \frac{12}{n(n+1)}.$$

*Proof.* Recall that

$$\lambda_2 = \min_{\vec{x} \perp \vec{1}} \frac{\vec{x}^T L_G \vec{x}}{\vec{x}^T \vec{x}}.$$

So for any vector $\vec{x}$ satisfying $\vec{x} \cdot \vec{1} = 0$, we have

$$\lambda_2 \vec{x}^T \vec{x} \leq \vec{x}^T L_G \vec{x}.$$

We can take the test vector $\vec{x}$ to be

$$\vec{x}(a) = (n+1) - 2a, \qquad \forall 1 \leq a \leq n.$$

Then

$$\lambda_2(P_n) \leq \frac{\sum_{a=1}^{n-1} (\vec{x}(a+1) - \vec{x}(a))^2}{\sum_{a=1}^{n} \vec{x}(a)^2} = \frac{4(n-1)}{\sum_{a=1}^{n} ((n+1) - 2a)^2} = \frac{12}{n(n+1)}.$$

For the lower bound, we first write $K_n = \sum_{i<j} G_{i,j}$. By Theorem 6.14, we have

$$G_{i,j} \preceq (j-i) \sum_{k=i}^{j-1} G_{k,k+1} \preceq (j-i) P_n.$$

So

$$K_n = \sum_{i<j} G_{i,j} \preceq (\sum_{i<j} (j-i)) P_n = \frac{n(n-1)(n+1)}{6} P_n.$$

By Theorem 6.13 , We have

$$\lambda_2(P_n) \geq \frac{6}{n(n-1)(n+1)}\lambda_2(K_n) = \frac{6}{(n-1)(n+1)}.$$

Where $\lambda_2(K_n) = n$. ∎

**Theorem 6.16.** *For a weighted path $P_n$ with weights $w_i = w(i, i+1)$,*

$$G_{1,n} \leq (\sum_{i=1}^{n-1} \frac{1}{w_i}) \sum_{i=1}^{n-1} w_i G_{i,i+1}.$$

*Proof.* Exercise. ∎

### 6.4 Cayley Graphs

Given a group $\Gamma$ and a subset $S \subseteq \Gamma$ (elements of S are called generaters) which is closed under inverse (that is: $s \in S$ if and only if $s^{-1} \in S$) , the Cayley graph $G(\Gamma, S)$ is a graph with vertex set $\Gamma$ , where $u \sim_G v$ if and only if there exists $s \in S$ such that $u \cdot s = v$.

**Example 6.17.** *1) cycles $C_n = G(\mathbf{Z}_n, \{1, -1\})$.*

*2) hypercube $Q_n$ is a Caylay graph over the additive group $(\frac{\mathbf{Z}}{2\mathbf{Z}})^d$ , where $S$ is the set of vectors in $\{0,1\}^d$ that have a unique 1 in it's entries.*

Next, we will consider a special kind of Caylay graphs called Paley graphs. Let $p$ be a prime, which equals 1 modulo 4. Let $\mathbf{F}_p$ be the field of size $p$. Let $S = S_p$ be the set of all elements $s \in \mathbf{F}_p \setminus \{0\}$ such that those exists an $x$ for which $x^2 \equiv s \; mod \; p$. Such $s \in S$ is called quadratic residues. Let the Paley graph of order $p$ be $G(\mathbf{F}_p, S_p)$.

**Lemma 6.18.** *Let $S \triangleq S_p = \{x \in \mathbb{F}_p \backslash \{0\} : \exists\, y, \text{such that}\; x = y^2 \mod p\}$. Then $S$ is closed under the addition of $\mathbb{F}_p$. That is $s \in S$,if and only if $-s \in S$.*

*Proof.* We will need some facts from finite field.
**Fact 1.** For any prime $p$,there exists some $g$ such that every $x \in \mathbb{F}_p \backslash \{0\}$ . There exists an unique $i \in \{1, \ldots, p-1\}$.Such that $x \equiv g^i \mod p$. In particular, $g^{p-1} \equiv 1$.
**Fact 2.** If $p \equiv 1 \mod 4$ is a prime, then $-1 \in S_p$.

*Proof of Fact 2.* Take $g$ from Fact 1, and let $s = g^{(p-1)/4}$. Now, we show $s^2 = -1 \mod p$. Note that $s^4 = g^{p-1} \equiv -1 \mod p$. Also consider the equation $x^2 - 1 \equiv 0 \mod p$. Since $\mathbb{F}_p$ is a field, the equation has at most 2 solutions. And we know two solutions $x = 1$ and $x = s^2$, where we also know that $s^2 \neq 1 \mod p$ (otherwise $g^{(p-1)/2} \equiv 1 \equiv g^{p-1}$, it is a contradiction.). Therefore, $s^2 = -1 \mod p$. ∎

Now, by Fact 1, we understand that the squares in $S$ are exactly the elements $g^i$ for even $i$. As $-1 \equiv (g^{(p-1)/4})^2 \mod p$. We see $g^i \in S$, if and only if $-g^i = g^i \cdot g^{2(p-1)/4} \in S$.

∎

**Proposition 1.** $|S_p| = \frac{(p-1)}{2}$*, and thus $G = (\mathbb{F}_p, S)$ is $\frac{(p-1)}{2}$-regular.*

**Theorem 6.19.** *The eigenvalues of Laplacian Matrix $L$ of a Paley graph of $\mathbb{F}_p$ are $0, \frac{(p-\sqrt{p})}{2}$ and $\frac{(p+\sqrt{p})}{2}$.*

**Lemma 6.20.** $L^2 = pL + \frac{(p-1)}{4}J - \frac{p(p-1)}{4}I$, *where $I$ is the identity matrix, and $J$ is the matrix which all entries are 1.*

*Proof.* Proof Theorem 6.19 by using Lemma 6.20.

Let $\lambda \neq 0$ be an eigenvalue of $L$, and let $\vec{x}$ be its eigenvector. Then$\vec{x}\vec{1} = 0, \vec{x}J = 0$. So,

$$L^2\vec{x} = pL\vec{x} - \frac{p(p-1)}{4}\vec{x},$$

which implies that

$$\lambda^2\vec{x} = p\lambda\vec{x} - \frac{p(p-1)}{4}\vec{x} \Rightarrow \lambda^2 = p\lambda - \frac{p(p-1)}{4} \Rightarrow \lambda = \frac{(p \pm \sqrt{p^2 - p(p-1)})}{2} = \frac{p \pm \sqrt{p}}{2}.$$

∎

**Definition 6.21.** *Let the quadratic character $\chi : \mathbb{F}_p \to \{0, 1, -1\}$ be given by* $\chi(x) = \begin{cases} 1 & \textit{if } x \in S_p \\ 0, & \textit{if } x = 0, \\ -1 & \textit{otherwise.} \end{cases}$

*Observe that $\chi(xy) = \chi(x)\chi(y)$.*

**Theorem 6.22.** *Let $X$ be a matrix, given by $X(u, v) = \chi(u-v)$. And* $X(u, v) = \begin{cases} 1 & \textit{if } (u-v) \in S \\ -1, & \textit{if } (u-v) \notin S. \end{cases}$

*So, we have* $\begin{cases} X = pI - 2L - J, \\ X^2 = pL - J. \end{cases}$

*Proof.* Let $X = (\vec{v_1}, \dots, \vec{v_p})$, so the diagonal entry of $X^2$ is $\vec{v_i}^T\vec{v_i} = \|\vec{v_i}\| = p - 1$. Because each $\vec{v_i}$ has exactly $\frac{p-1}{2}$ 1-erntries and $\frac{p-1}{2}$ (-1)-erntries. The off-diagonal entries of $X^2$ are the inner products of columns of $X$. That is

$$X^2(i, j) = \vec{v_i}\vec{v_j} = \sum_x \chi(i - x)\chi(j - x) = \sum_{y \in \mathbb{F}_p \setminus \{0\}} \chi(y)\chi((j - i) + y),$$

where $y = i - x, i \neq j$ .

Note that, $y \neq 0, \chi(y) \in \{1, -1\}$. So,$\chi(y)\chi((j-i)+y) = \chi(\frac{j-i+y}{y}) = \chi(\frac{j-i}{y} + 1)$. As $y$ varies over $\mathbb{F}_p \setminus \{0\}$. $\frac{j-i}{y}$ also varies over $\mathbb{F}_p \setminus \{0\}$. Thus $\frac{j-i}{y} + 1$ also varies over $\mathbb{F}_p \setminus \{1\}$.

Combining, $X^2(i, j) = \sum_{y \in \mathbb{F}_p \setminus \{0\}} \chi(\frac{j-i}{y} + 1) = \sum_{k \in \mathbb{F}_p \setminus \{1\}} \chi(k) = -1.$

∎

*Proof.* Proof of Lemma 6.20:

$$\begin{cases} X = pI - 2L - J \\ X^2 = pL - J \end{cases} \Longrightarrow L^2 = pL + \frac{(p-1)}{4}J - \frac{p(p-1)}{4}I.$$

∎

**Definition 6.23.** *We say $H$ is c-approximation of $G$, if and only if $cH \geq G \geq H/c$, i.e. $L_cH \geq L_G \geq L_H/c$.*

We are interested in finding the following graphs $G$ (called expander graph) which are "sparse" and there exists some "small" $\varepsilon \geq 0$ such that $(1+\varepsilon)K_n \geq G \geq K_n/(1+\varepsilon)$.

**Exercise 2.** *Paley graph $G = (\mathbb{F}_p, S_p)$ is $(1+\frac{1}{\sqrt{p}})$-approximation of the complete graph $K_p$.*

**Definition 6.24.** *The $n$-dimensional hypercube, $Q_n$, is the graph whose vertex set is $\{0,1\}^n$ and whose edge set is the set of pairs that differ in exactly one coordinate. So, $\vec{v} \in \{0,1\}^n \Rightarrow \vec{v} = -\vec{v}$.*

Any $\vec{g} = \{0,1\}^d$ satisfies $\vec{g} = -\vec{g}$. So, if we take any vectors $(\vec{g_1}, \ldots, \vec{g_k}) \in \{0,1\}^d$, this would define a Cayley graph $G(V, \{\vec{g_1}, \ldots, \vec{g_k}\})$, we call such graphs as generalizing hypercubes.
From now on, let $G = G(V, \{\vec{g_1}, \ldots, \vec{g_k}\})$.

**Definition 6.25.** *For $\vec{b} \in V$, define $\vec{\varphi}_{\vec{b}} : V \to R$, by $\vec{\varphi}_{\vec{b}}(\vec{x}) = (-1)^{\vec{b}\vec{x}}$.*

**Lemma 6.26.** *For each $\vec{b} \in V$, the vector $\vec{\varphi}_{\vec{b}}$ is an eigenvector of the Laplacian Matrix of the generalized hypercube $G$ with respect to the eigenvalue $k - \sum_{i=1}^{k}(-1)^{\vec{b}\vec{g_i}} = \lambda_{\vec{b}}$.*

*Proof.* Exercise. ∎

**Chernoff bound**: Let $x_1, \ldots, x_k$ be independent $\pm 1$ random variables. Then for all $t > 0$,

$$P_r(\left|\sum_{i=1} x_i\right| > t) \leq 2e^{-t^2/2k}.$$

**Lemma 6.27.** *Take $\vec{g_1}, \ldots, \vec{g_k} \in V$ independently and uniformly. Then with high probability, all of the nonzero eigenvalue of the generalized hypercube $G$ differ from $k$ by at most $k\sqrt{\frac{2}{c}}$, where $k = cd$.*

*Proof.* Recall that $\lambda_{\vec{b}} = k - \sum_{i=1}^{k}(-1)^{\vec{b}\vec{g_i}}$ is a eigenvalue of $G$. For fixed $\vec{b} \in V$, let $x_i = (-1)^{\vec{b}\vec{g_i}}$. Then $x_1, \ldots, x_k$ are independent $\pm 1$ random variables by Chernoff bound, for $t = k\sqrt{\frac{2}{c}}$. $P_r(|\lambda_{\vec{b}} - k|) = P_r(\left|\sum_{i=1} x_i\right| > t) \leq 2e^{-t^2/2k} \leq 2e^{k/c} = 2e^{-d}$ holds for any fixed $\vec{b} \in V$. By union bound, the probability that there is some $\vec{b}$ for which $|\lambda_{\vec{b}} - k| > t = t = k\sqrt{\frac{2}{c}}$ is at most $\sum_{\vec{b} \in V} P_r(|\lambda_{\vec{b}} - k|) \leq 2^d 2e^{-d} = 2\left(\frac{2}{e}\right)^d$, which is going to 0 as $d \to +\infty$. ∎

**Theorem 6.28.** *For Cayley graphs over Abelian groups one can construct an orthogonal basis of eigenvectors without even knowing the set of generators $S$(i.e. independent of $S$ ).*

**Theorem 6.29** (Hoffman's bound)**.** *Let $G$ be an $n$-vertex $d$-regular graphs and let $\mu_1 \geq \cdots \geq \mu_n$ be the eigenvalues of $A(G)$. Then, the size of the largest independent set in $G$ satisfies $\alpha(G) \leq n\frac{-\mu_n}{\mu_1 - \mu_n}$, where $\mu_1 = d$.*

**Theorem 6.30** (Godsil and Newman)**.** *Let $S$ be an independent set in $G$ and let $d_{ave}(S) = \sum_{v\in S} \frac{d_G(S)}{|S|}$ be the average degree of vertices in $S$. Then $|S| \le n(1 - \frac{d_{ave}(S)}{\lambda_n})$, where $\lambda_n$ is the largest eigenvalue of $L(G)$.*

*Proof.* Recall that $\lambda_n = \max_{\vec{x}} \frac{\vec{x}^T L \vec{x}}{\vec{x}^T \vec{x}}$. We prove that any vectors $\vec{x}$ maximizing that is the eigenvector of $\lambda_n$, which must be orthogonal to $\vec{1}$. So, $\lambda_n = \max_{\vec{x}\perp\vec{1}} \frac{\vec{x}^T L \vec{x}}{\vec{x}^T \vec{x}}$, consider $\vec{x}_0 = \vec{x}_s - s\vec{1}$, where $s = \frac{|S|}{n}$. As $S$ is independent, we have

$$\vec{x}_0^T L \vec{x}_0 = \sum_{i\sim j} (\vec{x}_0(i) - \vec{x}_0(j))^2 = \sum_{i\sim j} (\vec{x}_s(i) - \vec{x}_s(j))^2 = |\alpha S| = \sum_{v\in S} d(v) = d_{ave}(S)\,|S|\,.$$

Also, $\vec{x}_0^T \vec{x}_0 = n(s - s^2)$. Combining, we have

$$\lambda_n \ge \frac{\vec{x}^T L \vec{x}}{\vec{x}^T \vec{x}} = \frac{d_{ave}(S)\,|S|}{n(s - s^2)} = \frac{d_{ave}(S)}{1 - s}.$$

That is $1 - \frac{d_{ave}(S)}{\lambda_n} \ge s = \frac{|S|}{n}$. ∎

# 7 Godsil-Newman Theorem

**Theorem 7.1** (Godsil-Newman)**.** *Let $S$ be any independent set in $G$, and let $d_{ave}(S) = \sum_{v\in S} \frac{d_G(v)}{|S|}$. Then*

$$|S| \le n\left(1 - \frac{d_{ave}(S)}{\lambda_n}\right).$$

.

**Theorem 7.2** (Hoffman's bound)**.** *If $G$ is a $d$-regular graph, then*

$$\alpha(G) \le n\frac{-\mu_n}{d - \mu_n}.$$

**Definition 7.3.** *The chromatic number $\chi(G)$ of a graph $G$ is the smallest integer $k$ such that $V(G)$ can be partited into $k$ disjoint independent sets.*

**Theorem 7.4.** *For any graph $G$, $\chi(G) \ge 1 + \frac{\mu_1}{-\mu_n}$.*

**Lemma 7.5.** *Let $A$ be a symmetric with largest eigenvalue $\alpha_1$. Let $B$ be the matrix obtained from $A$ by removing the last row and column from $A$.*
*Let $\beta_1$ be the largest eigenvalue of $B$. Then $\alpha_1 \ge \beta_1$.*

*Proof.* Exercise. ∎

**Remark 7.6.** *Here one can take $G$ as a weighted graph with weight function $w : E(G) \to \mathbb{R}^+ \cup \{0\}$.*

**Lemma 7.7.** *Let* $A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1k} \\ A_{12}^T & A_{22} & \cdots & A_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1k}^T & A_{2k}^T & \cdots & A_{kk} \end{bmatrix}$ *be a block-partitioned symmetric matrix with* $k \geq 2$. *Then*

$$(k-1)\lambda_{min}(A) + \lambda_{max}(A) \leq \sum_{i=1}^{k} \lambda_{max}(A_{ii}).$$

*Proof.* We use induction on $k$. First consider $k = 2$. Then we write $A = \begin{bmatrix} B & C \\ C^T & D \end{bmatrix}$ and we need to show $\lambda_{min}(A) + \lambda_{max}(A) \leq \lambda_{max}(B) + \lambda_{max}(D)$. Take $\boldsymbol{x}$ to be a unit eigenvector of $A$ of the eigenvalue $\lambda_{max}(A)$. Write $\boldsymbol{x} = \begin{pmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{pmatrix}$. We consider two cases.

**Case 1.** Neither $\boldsymbol{x}_1$ nor $\boldsymbol{x}_2$ is a zero-vector.

Let $\boldsymbol{y} = \begin{pmatrix} \frac{||\boldsymbol{x}_2||}{||\boldsymbol{x}_1||}\boldsymbol{x}_1 \\ \\ \frac{||\boldsymbol{x}_1||}{||\boldsymbol{x}_2||}\boldsymbol{x}_2 \end{pmatrix}$. Then $\boldsymbol{y}$ is also a unit vector. By Courant-Fischer, we have

$$\lambda_{min}(A) \leq \boldsymbol{y}^T A \boldsymbol{y}.$$

So

$$\begin{aligned} \lambda_{max}(A) + \lambda_{min}(A) \leq & \boldsymbol{x}^T A \boldsymbol{x} + \boldsymbol{y}^T A \boldsymbol{y} \\ = & \boldsymbol{x}_1^T B \boldsymbol{x}_1 + \boldsymbol{x}_1^T C \boldsymbol{x}_2 + \boldsymbol{x}_2^T C \boldsymbol{x}_1 + \boldsymbol{x}_2^T D \boldsymbol{x}_2 + \\ & \frac{||\boldsymbol{x}_2||^2}{||\boldsymbol{x}_1||^2}\boldsymbol{x}_1^T B \boldsymbol{x}_1 - \boldsymbol{x}_1^T C \boldsymbol{x}_2 - \boldsymbol{x}_2^T C \boldsymbol{x}_1 + \frac{||\boldsymbol{x}_1||^2}{||\boldsymbol{x}_2||^2}\boldsymbol{x}_2^T D \boldsymbol{x}_2 \\ = & \left(1 + \frac{||\boldsymbol{x}_2||^2}{||\boldsymbol{x}_1||^2}\right)\boldsymbol{x}_1^T B \boldsymbol{x}_1 + \left(1 + \frac{||\boldsymbol{x}_1||^2}{||\boldsymbol{x}_2||^2}\right)\boldsymbol{x}_2^T D \boldsymbol{x}_2 \\ \leq & \lambda_{max}(B)(||\boldsymbol{x}_1||^2 + ||\boldsymbol{x}_2||^2) + \lambda_{max}(D)(||\boldsymbol{x}_1||^2 + ||\boldsymbol{x}_2||^2) \\ = & \lambda_{max}(B) + \lambda_{max}(D), \end{aligned}$$

as $\boldsymbol{x} = \begin{pmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{pmatrix}$ is a unit vector.

**Case 2.** Assume $\boldsymbol{x}_2 = \mathbf{0}$ (similar for $\boldsymbol{x}_1 = \mathbf{0}$).

By Lemma 7.5, $\lambda_{max}(B) \leq \lambda_{max}(A)$ and $\lambda_{min}(D) \geq \lambda_{min}(A)$. Since $\boldsymbol{x}_2 = \mathbf{0}$, $\boldsymbol{x}_1$ is also a unit vector. Further, $\lambda_{max}(B) \geq \boldsymbol{x}_1^T B \boldsymbol{x}_1 = \boldsymbol{x}^T A \boldsymbol{x} = \lambda_{max}(A)$, implying $\lambda_{max}(A) = \lambda_{max}(B)$. Thus, we have

$$\lambda_{min}(A) + \lambda_{max}(A) \leq \lambda_{min}(D) + \lambda_{max}(B) \leq \lambda_{max}(B) + \lambda_{max}(D),$$

done!

These prove the case $k = 2$.

For $k \geq 3$, we apply induction. Let $B = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1k} \\ A_{12}^T & A_{22} & \cdots & A_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1k}^T & A_{2k}^T & \cdots & A_{kk} \end{bmatrix}$ and $A = \begin{bmatrix} B & C \\ C^T & A_{kk} \end{bmatrix}$. By Lemma 7.5, $\lambda_{min}(B) \geq \lambda_{min}(A)$. Then as we proved the case $k = 2$,

$$\lambda_{min}(A) + \lambda_{max}(A) \leq \lambda_{max}(B) + \lambda_{max}(A_{kk}).$$

By induction on $B$,

$$(k-2)\lambda_{min}(B) + \lambda_{max}(B) \leq \sum_{i=1}^{k-1} \lambda_{max}(A_{ii}).$$

Combining,

$$(k-1)\lambda_{min}(A) + \lambda_{max}(A) \leq \sum_{i=1}^{k} \lambda_{max}(A_{ii}).$$

This proves Lemma 7.7. ∎

*Proof of Theorem 7.4.* Let $k = \chi(G)$. Then, after possibly re-ordering the vertices, the adjacency matrix $A(G)$ can be written as

$$A = \begin{bmatrix} O & A_{12} & \cdots & A_{1k} \\ A_{12}^T & O & \cdots & A_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1k}^T & A_{2k}^T & \cdots & O \end{bmatrix}.$$

By Lemma 7.7,

$$(k-1)\mu_n + \mu_1 \leq \sum_{i=1}^{k} \lambda_{max}(O) = 0.$$

$$\mu_1 - \mu_n \leq k(-\mu_n) \Rightarrow \chi(G) = k \geq \frac{\mu_1 - \mu_n}{-\mu_n} = 1 + \frac{\mu_1}{-\mu_n}.$$

∎

**Theorem 7.8.** $\chi(G) \leq 1 + \lfloor \mu_1 \rfloor$.

**Fact 7.9.** *$G$ is $d$-regular* $\Rightarrow \chi(G) \leq d+1$

*Proof of Theorem 7.8.* We prove this by induction on the number $n$ of vertices.
For $n = 1$, it is just trival.
Now assume this holds for all graphs on $n-1$ vertices. Consider on $n$-vertex graph $G$.
By an earlier result,

$$\overline{d}_{ave} \leq \mu_1 \leq \Delta(G).$$

There exists a vertex $v$, whose degree is at most $\lfloor \mu_1 \rfloor$. Consider $G - \{v\}$. By Lemma 7.5,

$$\lambda_{max}(G - \{v\}) \leq \mu_1.$$

By induction on $G - \{v\}$,

$$\chi(G - \{v\}) \leq 1 + \lfloor \mu_1 \rfloor.$$

Since $v$ has at most $\lfloor \mu_1 \rfloor$ neighbors in $G - \{v\}$, we can extend a $(1 + \lfloor \mu_1 \rfloor)$-coloring of $G - \{v\}$ into a $(1 + \lfloor \mu_1 \rfloor)$-coloring of $G$. ∎

**Theorem 7.10** (Perron-Frobenius, symmetric)**.** *Let $G$ be a connected weighted graph. Then*

*(i)* $\mu_1 \geq -\mu_n$.

*(ii)* $\mu_1 > \mu_2 \iff \mu_1$ *has multiplicity* 1.

*(iii) The eigenvalue $\mu_1$ has a strictly positive eigenvector.*

*Proof.* The items (i) and (iii) were proved. We consider item (ii).
Consider an eigenvector $\boldsymbol{\varphi}_i$ of the eigenvalue $\mu_2$. By (iii), there exists an eigenvector $\boldsymbol{\varphi}_1$ of the eigenvalue $\mu_1$, which is strictly positive. We know $\boldsymbol{\varphi}_1 \perp \boldsymbol{\varphi}_2$. So $\boldsymbol{\varphi}_2$ contains both positive and negative entries. Let $\boldsymbol{y}$ be a vector such that $\boldsymbol{y}(u) = |\boldsymbol{\varphi}_2(u)|$ for $\forall u$. Then

$$\mu_2 = \frac{\boldsymbol{\varphi}_2^T A \boldsymbol{\varphi}_2}{\boldsymbol{\varphi}_2^T \boldsymbol{\varphi}_2} \leq \frac{\boldsymbol{y}^T A \boldsymbol{y}}{\boldsymbol{y}^T \boldsymbol{y}} \leq \mu_1.$$

The first " $\leq$ " should be strict " $<$ ". The reason is as following:
**Case 1.** $\boldsymbol{y}$ has a zero-entry. If $\mu_2 = \mu_1$, then $\boldsymbol{y}$ is an eigenvector of $\mu_1$. By a lemma before, $\boldsymbol{y} > \mathbf{0}$, a contradiction.
**Case 2.** $\boldsymbol{y}$ has no zero-entry. Since $G$ is connected, there must be some edge $(u, v)$ for which $\boldsymbol{\varphi}_2(u) < 0 < \boldsymbol{\varphi}_2(v)$. This edge will also show that $\boldsymbol{\varphi}_2^T A \boldsymbol{\varphi}_2 < \boldsymbol{y}^T A \boldsymbol{y}$, thus we have $\mu_1 < \mu_2$. ∎

# 8 Random Walk

## 8.1 Definition

A **random walk** on a weighted graph $G$ with $w : E(G) \to \mathbb{R}^+$, is a process that begins at some vertex, and at each time(step) moves to another vertex(a neighbor of the current vertex) with probability proportional to the weight of the corresponding edge. For example, if now we are at vertex $a \in V(G)$, then we move to $b \in N(a)$ with probability $p_{ab} = w(ab)/(\sum_{u \in N(a)} w(au))$.

A **probability vector** $\vec{p}$ is a vector in $\mathbb{R}^{V(G)}$ such that $\vec{p}(a) \geq 0$ for all $a \in V(G)$ and $\sum_{a \in V(G)} \vec{p}(a) = 1$.

We will let $\vec{p}_t$ denote the **probability distribution** at time $t$. Let $\vec{d}$ be a vector with $\vec{d}(a) = \sum_{ab \in E(G)} w(ab)$ for all $a \in V(G)$. One can state the relationship as

$$\forall a \in V(G), \forall t \in \mathbb{N}, \vec{p}_{t+1}(a) = \sum_{ab \in E(G)} \frac{w(ab)}{\vec{d}(b)} \vec{p}_t(b). \tag{8.2}$$

Let $M$ be a matrix with $(M)_{x,y} = \begin{cases} w(xy), & xy \in E(G), \\ 0, & \text{otherwise.} \end{cases}$
Then (8.2) becomes $\vec{p}_{t+1} = M D^{-1} \vec{p}_t$, where $D = diag(\vec{d}(1), \vec{d}(2), \ldots, \vec{d}(n))$ is a diagonal matrix.

We consider **lazy random walks**, which are the variant of a random walk that stay put with probability $\frac{1}{2}$ and walk to a random neighbor with probability $\frac{1}{2}$.

Then these evolve according to the equation

$$\forall t \in \mathbb{N}, \vec{p}_{t+1} = \frac{1}{2}\vec{p}_t + \frac{1}{2} M D^{-1} \vec{p}_t = \frac{1}{2}(I + M D^{-1})\vec{p}_t.$$

Let $W_G = \frac{1}{2}(I + M D^{-1})$ be the **lazy walk matrix** of the weighted graph $G$.

Let $N = D^{-\frac{1}{2}}LD^{-\frac{1}{2}}$ be the **normalized Laplacian matrix** of $G$, where $L = D - M$.

**Remark 8.1.** *$D^{-\frac{1}{2}} = diag(\vec{d}(1)^{-\frac{1}{2}}, \vec{d}(2)^{-\frac{1}{2}}, \ldots, \vec{d}(n)^{-\frac{1}{2}})$. We often use this without definition for diagonal matrixs.*

**Remark 8.2.** *We can also write $N = I - D^{-\frac{1}{2}}MD^{-\frac{1}{2}}$.*

## 8.2 The Convergence of Lazy Random Walk

**Proposition 8.3.** *$W_G = I - \frac{1}{2}D^{\frac{1}{2}}ND^{-\frac{1}{2}}$.*

*Proof.*

$$\begin{aligned} W_G &= \frac{1}{2}(I + MD^{-1}) = I - \frac{1}{2}(I - MD^{-1}) \\ &= I - \frac{1}{2}D^{\frac{1}{2}}(I - D^{-\frac{1}{2}}MD^{-\frac{1}{2}})D^{-\frac{1}{2}} = I - \frac{1}{2}D^{\frac{1}{2}}ND^{-\frac{1}{2}}. \end{aligned}$$

∎

**Proposition 8.4.** *For every eigenvector $\vec{\varphi}_i$ of $N$ with eigenvalue $\nu_i$, $D^{\frac{1}{2}}\vec{\varphi}_i$ is the eigenvector of $W_G$ with eigenvalue $1 - \frac{\nu_i}{2}$.*

*Proof.* As $N\vec{\varphi}_i = \nu_i\vec{\varphi}_i$, by Proposition 8.3, we have

$$W_G(D^{\frac{1}{2}}\vec{\varphi}_i) = D^{\frac{1}{2}}\vec{\varphi}_i - \frac{1}{2}D^{\frac{1}{2}}ND^{-\frac{1}{2}} \cdot D^{\frac{1}{2}}\vec{\varphi}_i = (1 - \frac{\nu_i}{2})D^{\frac{1}{2}}\vec{\varphi}_i.$$

∎

**Proposition 8.5.** *Let $0 = \nu_1 \leq \nu_2 \leq \cdots \leq \nu_n$ be the eigenvalues of $N$, then for every $i$, $0 \leq \nu_i \leq 2$.*

*Proof.* Exercise.(Using $D \succeq -M$) ∎

**Theorem 8.6.** *Let $G$ be a connected weighted graph. Let $\vec{\pi} = \frac{\vec{d}}{\vec{1}^T\vec{d}}$, where $\vec{d}$ is the vector of (weighted) degree. Then the lazy random walk converges to $\vec{\pi}$, that is, $lim_{t\to\infty}\vec{p}_t = \vec{\pi}$.*

Let the eigenvalues of $W_G$ be $1 = \omega_1 \geq \omega_2 \geq \cdots \geq \omega_n \geq 0$, where $\omega_i = 1 - \frac{\nu_i}{2}$.

To prove Theorem 8.6, we need the following proposition.

**Proposition 8.7.** *If $G$ is connected, then $\omega_1 > \omega_2$.*

To prove Proposition 8.7, it suffices to show the following lemma.

**Lemma 8.8.** *Let $X$ be a symmetric matrix with non-positive off-diagonal entries such that the graph induced by the non-zero off-diagonal entries is connected. Let $\lambda_1$ be the smallest eigenvalue of $X$ and let $\vec{v}_1$ be the corresponding vector. Then $\vec{v}_1$ can be taken to be strictly positive and $\lambda_1$ has multiplicity 1.*

**Exercise 3.** *Prove Lemma 8.8.*

**Exercise 4.** *Check that $N$ satisfies the condition in Lemma 8.8.*

Now we give the proof of Theorem 8.6.

*Proof.* Let $\vec{\varphi}_1, \vec{\varphi}_2, \ldots, \vec{\varphi}_n$ be the orthonormal eigenvectors of $N$, corr-corresponding to the eigenvalues $0 = \nu_1 \le \nu_2 \le \cdots \le \nu_n$. Let $D^{-\frac{1}{2}}\vec{p}_0 = \sum c_i \vec{\varphi}_i$. We point out that $\vec{\varphi}_1 = \vec{d}^{\frac{1}{2}}/||\vec{d}^{\frac{1}{2}}||$. Then

$$c_1 = \vec{\varphi}_1^T (D^{-\frac{1}{2}}\vec{p}_0) = \frac{(\vec{d}^{\frac{1}{2}})^T}{||\vec{d}^{\frac{1}{2}}||} \cdot D^{-\frac{1}{2}}\vec{p}_0 = \frac{\vec{1}^T \cdot \vec{p}_0}{||\vec{d}^{\frac{1}{2}}||} = \frac{1}{||\vec{d}^{\frac{1}{2}}||}.$$

Also

$$c_1 D^{\frac{1}{2}} \vec{\varphi}_1 = \frac{1}{||\vec{d}^{\frac{1}{2}}||} \cdot D^{\frac{1}{2}} \cdot \frac{(\vec{d}^{\frac{1}{2}})^T}{||\vec{d}^{\frac{1}{2}}||} = \frac{\vec{d}}{||\vec{d}^{\frac{1}{2}}||^2} = \vec{\pi}.$$

Recall that $W_G = I - \frac{1}{2} D^{\frac{1}{2}} N D^{-\frac{1}{2}}$, and that $\vec{p}_t = W_G^t \vec{p}_0$. We have

$$\begin{aligned}
\vec{p}_t = W_G^t \vec{p}_0 &= (D^{\frac{1}{2}}(I - \frac{1}{2}N)D^{-\frac{1}{2}})^t \vec{p}_0 = D^{\frac{1}{2}}(I - \frac{1}{2}N)^t (D^{-\frac{1}{2}}\vec{p}_0) \\
&= D^{\frac{1}{2}}(I - \frac{1}{2}N)^t (\sum_{i=1}^n c_i \vec{\varphi}_i) = D^{\frac{1}{2}} \sum_{i=1}^n c_i (1 - \frac{\nu_i}{2})^t \vec{\varphi}_i \\
&= c_1 D^{\frac{1}{2}} \vec{\varphi}_1 + \sum_{i=2}^n c_i \omega_i^t D^{\frac{1}{2}} \vec{\varphi}_i = \vec{\pi} + \sum_{i=2}^n c_i \omega_i^t D^{\frac{1}{2}} \vec{\varphi}_i.
\end{aligned}$$

By Proposition 8.7, $\forall i \ge 2$, $\omega_i < 1$. So $\vec{p}_t \to \vec{\pi}$ as $t \to \infty$. ∎

**Theorem 8.9.** *For all $a, b \in V(G)$ and all $t \in \mathbb{N}$, if $\vec{p}_0 = \vec{\delta}_a$, then*

$$|\vec{p}_t(b) - \vec{\pi}(b)| \le \sqrt{\frac{\vec{d}(b)}{\vec{d}(a)}} \omega_2^t.$$

**Lemma 8.10.** *Suppose $L, N$ are the Laplacian matrix and normalized Laplacian matrix of graph $G$. Let $0 = \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$ be the eigenvalues of $L$, and $0 = \nu_1 \le \nu_2 \le \cdots \le \nu_n$ be the eigenvalues of $N$. Then for all $1 \le i \le n$, $\frac{\lambda_i}{\Delta(G)} \le \nu_i \le \frac{\lambda_i}{\delta(G)}$.*

*Proof.* By **Courant-Fischer's Theorem**,

$$\nu_i = \min_{dim(S)=i} \max_{\vec{x} \in S} \frac{\vec{x}^T N \vec{x}}{\vec{x}^T \vec{x}}.$$

Let $\vec{y} = D^{-\frac{1}{2}}\vec{x}$, then $\vec{x} = D^{\frac{1}{2}}\vec{y}$. So we have

$$\nu_i = \min_{dim(S)=i} \max_{\vec{y} \in S} \frac{\vec{y}^T D^{\frac{1}{2}} N D^{\frac{1}{2}} \vec{y}}{\vec{y}^T D \vec{y}} = \nu_i = \min_{dim(S)=i} \max_{\vec{y} \in S} \frac{\vec{y}^T L \vec{y}}{\vec{y}^T D \vec{y}}.$$

Since for every $\vec{y} \neq \vec{0}$

$$\frac{\vec{y}^T L \vec{y}}{\Delta(G) \vec{y}^T \vec{y}} \le \frac{\vec{y}^T L \vec{y}}{\vec{y}^T D \vec{y}} \le \frac{\vec{y}^T L \vec{y}}{\delta(G) \vec{y}^T \vec{y}}.$$

We finally have

$$\frac{\lambda_i}{\Delta(G)} \le \nu_i \le \frac{\lambda_i}{\delta(G)}.$$

∎

**Proposition 8.11.**

$$\nu_2 = \min_{\vec{x} \perp \vec{d}} \frac{\vec{x}^T L \vec{x}}{\vec{x}^T D \vec{x}}.$$

**Exercise 5.** *Prove Proposition 8.11.*

**Remark 8.12.** *Proposition 8.11 also works for weighted graphs, and it gives a way to upper-bound $\nu_2$.*

**Lemma 8.13.** *Let $G$ be an **unweighted** graph with diameter at most $r$, then $\lambda_2(G) \ge \frac{2}{r(n-1)}$.*

*Proof.* For any pair of vertices $\{a, b\}$, let $P(a, b)$ be a path in $G$ connecting $a$ and $b$ with length at most $r$. Then

$$L_{ab} \preceq rP(a, b) \preceq rL_G.$$

So

$$K_n = \sum_{\{a,b\}} ab \preceq \sum_{\{a,b\}} rG = r\binom{n}{2}G.$$

Then

$$n = \lambda_2(K_n) \le r\binom{n}{2}\lambda_2(G).$$

Finally we have $\lambda_2(G) \ge \frac{2}{r(n-1)}$. ∎

**Exercise 6.** *Let $D_n$ be a graph consisting of 2 $K_n$s joined by an edge. Prove that $\nu_2(D_n) = \Theta(\frac{1}{n^2})$.*

# 9 Cheeger's Inequality

**Definition 9.1.** *We define the conductance of a subset $S$ in $G$ to be*

$$\phi(S) = \frac{\partial S}{\min\{d(S), d(V \setminus S)\}},$$

*where*

$$d(S) = \sum_{x \in S} d_S(S).$$

**Definition 9.2.** *The conductance of the graph $G$ is*

$$\phi_G = \min_{S \subseteq V(G)} \phi(S).$$

*Earlier, we considered $\theta(S) = \frac{|\partial S|}{|S|}$ and $\theta_G = \min_{|S| \le \frac{M}{2}} \theta(S)$.*

**Theorem 9.3.** *$\theta_G \geq \frac{\lambda_2}{2}$, where $\lambda_2$ is the second smallest eigenvalue of $L(G)$.*

**Theorem 9.4.** *$\phi_G \geq \frac{\nu_2}{2}$, where $\nu_2$ is second smallest eigenvalue of $N = N(G) = D^{-\frac{1}{2}} L D^{-\frac{1}{2}}$.*

*Proof.* Omit, using $\nu_2 = \min_{\vec{x} \perp \vec{d}} \frac{\vec{x}^T L \vec{x}}{\vec{x}^T D \vec{x}}$. (Exercise.) ∎

**Theorem 9.5.** *$(Cheeger's\ inequality)$ $\phi_G \leq (2\nu_2)^{\frac{1}{2}}$, for any (weighted) graph $G$.*

We actually will prove a stronger result.

**Theorem 9.6.** *Let $\vec{y}$ be a vector orthogonal to $\vec{d}$. Let $S_t = \{u : \vec{y}(u) < t\}$ for $t \in \mathbf{R}$. Then, there is a real $t$ for which $S_t$ satisfied $\phi(S_t) \leq (2\frac{\vec{x}^T L \vec{x}}{\vec{x}^T D \vec{x}})^{\frac{1}{2}}$.*

**Lemma 9.7.** *Fix $\vec{y}$, let $\vec{v}_s = \vec{y} + z\vec{1}$. Then the minimum of $\vec{v}_z^T D \vec{v}_z$ is achieved at the $z$ for which $\vec{v}_z^T \vec{d} = 0$.*

*Proof.* Let $f(z) = \vec{v}_z^T D \vec{v}_z \implies \frac{df}{dz}(z) = 2\vec{v}_z^T \vec{d} = 0$. (Why? Exercise.) ∎

*Proof of Theorem 9.6.* Fix $\vec{y}$ with $\vec{y} \perp \vec{d}$. Let $\rho = \frac{\vec{y}^T L \vec{y}}{\vec{y}^T D \vec{y}}$, we want to show that there exists $t \in \mathbf{R}$, $\phi(S_t) \leq (2\rho)^{\frac{1}{2}}$. We assume that $\vec{y}(1) \leq \vec{y}(2) \leq \ldots \leq \vec{y}(n)$.

Let $\vec{z} = \vec{y} - \vec{y}(j)\vec{1}$. We see $\vec{z}^T L \vec{z} = \vec{y}^T L \vec{y}$. and by Lemma 9.7, $\vec{z}^T D \vec{z} \geq \vec{y}^T D \vec{y}$ as $\vec{y} \perp \vec{d}$. Then, we have $\frac{\vec{z}^T L \vec{z}}{\vec{z}^T D \vec{z}} \leq \rho$. We also normalize $\vec{z}$ so that $\vec{z}(1)^2 + \vec{z}(n)^2 = 1$, where $\vec{z}(1) \leq \vec{z}(2) \leq \ldots \leq \vec{z}(n)$.

From now on, set
$$S_t = \{u : \vec{z}(u) < t\},$$
we will choose $t \in [\vec{z}(1), \vec{z}(n)]$, with probability density $2|t|$, that is
$$P_r(t \in [a,b]) = \int_{t=a}^{b} 2|t|dt \qquad .$$

Observe that the total probability is 1, as
$$\int_{t=\vec{z}(1)}^{\vec{z}(n)} 2|t|dt = \int_{t=\vec{z}(1)}^{0} 2|t|dt + \int_{t=0}^{\vec{z}(n)} 2|t|dt = \vec{z}(1)^2 + \vec{z}(n)^2 = 1.$$

We will show
$$E[|\partial(S_t)|] \leq \sqrt{2\rho} E[\min\{d(S_t), d(V \setminus S_t)\}].$$

This imply that there exists some $t$ so that
$$|\partial(S_t)| \leq \sqrt{2\rho} \min\{d(S_t), d(V \setminus S_t)\}.$$

**Lemma 9.8.**
$$E[|\partial(S_t)|] = \sum_{uv \in E(G)} P_r(uv \in \partial(S_t)) \leq \sum_{uv \in E(G)} |\vec{z}(u) - \vec{z}(v)|(|\vec{z}(u)| + |\vec{z}(v)|).$$

*Proof.* An edge $uv$ with $\vec{z}(u) \leq \vec{z}(v)$ is in $\partial(S_t)$ if and only if $\vec{z}(u) \leq t \leq \vec{z}(v)$. This event occurs with probability
$$\int_{t=|\vec{z}(u)|}^{\vec{z}(v)} 2|t|dt = \begin{cases} |\vec{z}(u)^2 - \vec{z}(v)^2| & \text{if } sgn(\vec{z}(u)) = sgn(\vec{z}(v)), \\ \vec{z}(u)^2 + \vec{z}(v)^2 & \text{otherwise.} \end{cases}$$

This finishes the proof. ∎

**Lemma 9.9.**

$$\mathbb{E}[\min\{d(S_t), d(V \setminus S_t)\}] = \vec{z}^T D\vec{z}.$$

*Proof.* Note that

$$\mathbb{E}_t[d(S_t)] = \sum_{u\in V} P_r(u \in S_t)d(u) = \sum_{u\in V} P_r(\vec{z}(u) \leq t)d(u).$$

Recall the choice of $j$, so we have

$$\begin{cases} t \geq 0 \Longrightarrow d(V \setminus S_t) = \min\{d(S_t), d(V \setminus S_t)\}, \\ t < 0 \Longrightarrow d(S_t) = \min\{d(S_t), d(V \setminus S_t)\}. \end{cases}$$

Therefore, $u$ is in the smaller set if $t < 0$ and $u < j$, or $t \geq 0$ and $u \geq j$. Then we have,

$$\mathbb{E}_t[\min\{d(S_t), d(V \setminus S_t)\}] = \sum_{u<j} P_r[\vec{z}(u) < t < 0]d(u) + \sum_{u\geq j} P_r[\vec{z}(u) \geq t \geq 0]d(u).$$

$$= \sum_{u<j} \vec{z}(u)^2 d(u) + \sum_{u\geq j} \vec{z}(u)^2 d(u) = \vec{z}^T D\vec{z}.$$

By Lemma 9.8,

$$\mathbb{E}_t[d(S_t)] \leq \sum_{uv\in E(G)} |\vec{z}(u) - \vec{z}(v)|\ (|\vec{z}(u)| + |\vec{z}(v)|).$$

$$\leq \sqrt{\sum_{uv\in E(G)} (\vec{z}(u) - \vec{z}(v))^2} \sqrt{\sum_{uv\in E(G)} (|\vec{z}(u)| + |\vec{z}(v)|)^2}.$$

As $\vec{z}^T L\vec{z} \leq \rho\vec{z}^T D\vec{z}$, we see

$$\sum_{uv\in E(G)} (\vec{z}(u) - \vec{z}(v))^2 \leq \sum_{u\in V} \vec{z}(u)^2 d(u).$$

And

$$\sum_{uv\in E(G)} (|\vec{z}(u)| + |\vec{z}(v)|)^2 \leq 2 \sum_{uv\in E(G)} [\vec{z}(u)^2 + \vec{z}(v)^2] = 2\sum_{u\in V} \vec{z}(u)^2 d(u),$$

which implies that

$$\mathbb{E}[|\partial(S_t)|] \leq \sqrt{2\rho} \sum_{u\in V} \vec{z}(u)^2 d(u) = \sqrt{2\rho}E[\min\{d(S_t), d(V \setminus S_t)\}].$$

∎

This proved Theorem 9.6. ∎

**Exercise 7.** *Verify that this proof also works for weighted graphs,* $\omega : E(G) \longrightarrow \mathbf{R}^+ \cup \{0\}$.

## 10 Alon Boppana Lower Bound

**Theorem 10.1** (Alon-Boppana)**.** *There exists constant c such that for every connected $d-$regular graph G, $\lambda_2(G) \geq 2\sqrt{d-1}(1-\frac{c}{\Delta^2})$, where $\Delta$ is the diameter of G.*

**Corollary 10.2.** *Let $\{G_m\}_{m=1}^{+\infty}$ be a family of connected d-regular graphs, with $|V(G_m)| \to \infty$ as $m \to \infty$. Then we have:*

$$\liminf_{m\to\infty} \lambda_2(G_m) \geqslant 2\sqrt{d-1}.$$

*proof of Corollary 10.2.* $d^{\Delta} \geqslant |V(G_m)|$, hence $\Delta \to \infty$ as $m \to \infty$. So RHS on Theorem 10.1 inequality tends to $2\sqrt{d-1}$. ∎

**Question 10.3.** *Is there a family of d-regular graphs such that*

- *Number of vertices tends to $\infty$.*
- $\lambda_2(G) \leqslant 2\sqrt{d-1}$.

**Definition 10.4.** *An $(n,d)$-graph (n vertices d-regular) G is called Ramanujan if*

$$\lambda(G) = \max_{|\lambda|\neq d,\ \lambda \text{ is a eigenvalue of } G} |\lambda| \leqslant 2\sqrt{d-1}.$$

**Remark 10.5.** *G is bipatite implies that $\lambda(G) = \lambda_2$. Otherwise, $\lambda(G) = \max\{\lambda_2, -\lambda_n\}$.*

**Definition 10.6.** *Let $T_{d,k}$ be d-"regular" tree (non-leaf vertices have degree d) of height k.*

**Definition 10.7.** *We say a function $f : V \to \mathbb{C}$ is spherical around v if $f(u)$ depends only on $d_G(u,v)$.For any function $f : V \to \mathbb{C}$, we can define its spherical symmetrical around v to be*

$$\tilde{f}(u) = (\sum_{d_G(u',v)=d_G(u,v)} f(u'))/|\{u' \in V | d_G(u',v) = d_G(u,v)\}|.$$

Let $v$ be the root of $T_{d,k}$. If $\vec{g}$ is an eigenvector of $T_{d,k}$ write regard with $\lambda$, then $\tilde{g}$ is also an eigenvector of $T_{d,k}$ write regard with $\lambda$. So we may assume that $\vec{g}$ is spherical around $v$.

Let $g_i = \vec{g}(u)$ where $d_G(u,v) = i$. Then we have

$$\begin{cases} \lambda g_0 = d g_1, \\ \lambda g_i = g_{i-1} + (d-1)g_{i+1}, i \in [k], \\ g_{k+1} = 0. \end{cases} \tag{10.3}$$

(To simplify, we assume there is a $(k+1)$-level whose values are 0.)

**Lemma 10.8.** *$\vec{g}$ is an eigenvector of $T_{d,k}$ write regard with $\lambda$ as above, then we have*

1. $|\lambda| < 2\sqrt{d-1}$.
2. *There is an $\lambda > (1-\frac{c}{\Delta^2})2\sqrt{d-1}$ (with $c \simeq 2\pi^2$) such that there is a real solution g of* (10.3) *that is non-negative and non-increasing.*

*Proof.* First we will proof (1). Let $i$ be the index such that $g_i(d-1)^{\frac{i}{2}}$ is maximum.
**Case 1.** if $i = 0$,

$$|\lambda| = |\frac{dg_1}{g_0}| \leqslant |\frac{dg_1}{\sqrt{d-1}g_1}| < 2\sqrt{d-1}.$$

**Case 2.** if $i \neq 0$,

$$\begin{aligned}|\lambda| &= |\frac{g_{i-1} + (d-1)g_{i+1}}{g_i}| \leqslant |\frac{g_{i-1}}{g_i}| + (d-1)|\frac{g_{i+1}}{g_i}| \\ &\leqslant |\frac{\sqrt{d-1}g_i}{g_i}| + (d-1)|\frac{g_{i+1}}{g_{i+1}\sqrt{d-1}}| = 2\sqrt{d-1}.\end{aligned}$$

The equality holds if and only if

$$\forall\ i, j,\ \ g_i(d-1)^{i/2} = g_j(d-1)^{j/2},$$

which is a contradiction with $g_{k+1} = 0$, so (1) holds.

To prove (2), we define $h : [k+1] \cup \{0\} \to \mathbb{R}$ by

$$\begin{cases} h_i = (d-1)^{-\frac{i}{2}} \sin((k+1-i)\theta) \ (\theta \text{ to be determined}). \\ h_{k+1} = (d-1)^{-\frac{k+1}{2}} \sin 0 = 0. \end{cases}$$

Then we have

$$\begin{aligned} h_{i-1} + (d-1)h_{i+1} &= (d-1)^{-\frac{i-1}{2}}[\sin((k+2-i)\theta) + \sin((k-i)\theta)] \\ &= 2\sqrt{d-1}(d-1)^{-i/2}\sin((k+1-i)\theta)\cos\theta \\ &= 2\sqrt{d-1}\cos\theta h_i. \end{aligned}$$

Let $\lambda = 2\sqrt{d-1}\cos\theta$. If we want $h = g$, by equation (10.3), we have$2\sqrt{d-1}\cos\theta(d-1)^{-0/2}\sin((k+1-0)\theta) = d(d-1)^{-1/2}\sin(k\theta)$.

Let $f(\theta) = 2(d-1)\cos\theta\sin(k\theta+\theta) - dsin(k\theta)$. Then we have $f(\frac{\pi}{k+1}) = -dsin(\frac{k\pi}{k+1}) < 0$, and

$$\lim_{\theta\to 0}\frac{2(d-1)\cos\theta\sin(k\theta+\theta)}{dsink\theta} = \frac{2(d-1)(k+1)}{dk} > 1.$$

It implies that $f(0^+) > 0$. So there exits $\theta_0 \in (0, \frac{\pi}{k+1})$ such that $f(\theta_0) = 0$ (smallest positive root). We know $\frac{\pi}{k+1} \simeq \frac{2\pi}{\Delta}$, by Taylor expansion we have

$$\cos\theta_0 > 1 - \frac{c}{\Delta^2},$$

thus we have

$$\lambda = 2\sqrt{d-1}\cos\theta > (1 - \frac{c}{\Delta^2})2\sqrt{d-1}.$$

It is easy to check $h$ is Non-negative and Non-decreasing. ∎

*Proof of Theorem 10.1.* Consider $s, t \in V(G)$ satisfying $d_G(s,t) = \Delta$. Let $k = \lfloor\frac{\Delta}{2}\rfloor - 1$.

By Courant-Fischer Theorem, we have

$$\lambda_2(G) = \max_{f\perp\mathbf{1}}\frac{f^TA_Gf}{f^Tf}.$$

Let $S_i = \{v \in V(G) | d_G(s,v) = i\}, T_i = \{v \in V(G) | d_G(t,v) = i\}, Q = V(G) \backslash ((\bigcup_{0 \leqslant i \leqslant k} S_i) \bigcup (\bigcup_{0 \leqslant i \leqslant k} T_i))$. Notice that, for any $0 \leqslant i, j \leqslant k,\ S_i \bigcap T_j = \emptyset$. By lemma 10.8, there exist an eigenvector $\vec{g}$ write regard with $\lambda$ of $T_{d,k}$ such that $\lambda > (1 - \frac{c}{\Delta^2}) 2\sqrt{d-1}$ (with $c \simeq 2\pi^2$) and $g$ is non-negative and non-increasing. Define $f : V(G) \cup \{0\} \to \mathbb{R}$ by

$$f(v) = \begin{cases} c_1 = g_i \quad if\ v \in S_i, \\ -c_2 = g_i \quad if\ v \in T_i, \\ 0 \quad if\ v \in Q. \end{cases}$$

By choosing suitable $c_1$ and $c_2$ such that

$$\sum_{v \in \bigcup S_i} f(v) = - \sum_{v \in \bigcup T_i} f(v),$$

which implies that $f \perp \mathbf{1}$, so we only need to prove $\frac{f^T A_G f}{f^T f} \leqslant \lambda$.

**Claim 3.** *(1)* $(Af)_v \leqslant \lambda f_v,\ if\ v \in \bigcup_i S_i$; *(2)* $(Af)_v \geqslant \lambda f_v,\ if\ v \in \bigcup_i T_i$.

By symmetry, we will only verify (1).

Let $v \in S_i$ for some $i > 0$. Among its neighbors, assume $|S_{i-1} \bigcap N_G(v)| = p,\ |S_i \bigcap N_G(v)| = q,\ |S_{i+1} \bigcap N_G(v)| = d - p - q,\ p \leqslant 1$. Then we have

$$\begin{aligned} (Af)_v &= c_1(pg_{i-1} + qg_i + (d-p-q)g_{i+1} \\ &\leqslant c_1(g_{i-1} + (d-1)g_{i+1}) \quad (\{g_i\}\ \textit{non-increasing}) \\ &= c_1 \lambda g_i = \lambda f_v, \end{aligned}$$

Which complete the proof of the claim.

By the above claim, we have

$$\begin{aligned} f^T A f &= \sum_{v \in \bigcup S_i} f_v^T (Af)_v + \sum_{v \in \bigcup T_i} f_v^T (Af)_v \\ &\leqslant \sum_{v \in \bigcup S_i} \lambda f_v^T f + \sum_{v \in \bigcup T_i} \lambda f_v^T f \\ &= \lambda f^T f. \end{aligned}$$

. ∎

# 11 Bipartite Ramanujan Graph

## 11.1 2-Lift

**Definition 11.1.** *Given a graph $G = (V, E)$, a 2-lift of $G$ is a graph $\widehat{G}(\widehat{V}, \widehat{E})$ that has two vertices $\{v_0, v_1\} \in \widehat{V}$ for each $v \in V$. This pair of vertices is called the fiber of original vertex, every edge in $G$ corresponds to two edges in $\widehat{G}$.*

$$u \leftrightarrow \{u_0, u_1\} \quad v \leftrightarrow \{v_0, v_1\}.$$

$\widehat{E}$ *can either contains (1)* $\{(u_0, v_0\}, \{u_1, v_1\}$ *(2)* $\{(u_0, v_1\}, \{u_1, v_0\}$.

**Definition 11.2.** *Define a sign* $S : E \to \{\pm 1\}$. $S(u, v) = 1$ *if u,v satisfy type (1),* $S(u, v) = -1$ *otherwise. this defines a matrix* $A_s$ *just like the adjacent matrix.*

**Lemma 11.3.** *The eigenvalue of the 2-lift are the union, taken with multiplicity of the eigenvalues of* $A$ *and* $A_s$.

*Proof.* Denote $A_1$ as the positive part of $A_s$ and $A_{-1}$ as the negative part. Let $\widehat{A}$ be the adjacent matrix of $\widehat{G}$.

$$\widehat{A} = \begin{pmatrix} & A_1 - A_{-1} \\ -A_1 & A_1 \end{pmatrix}.$$

Let $x$ be an eigenvector of A with eigenvalue $\lambda$ , then we have

$$\widehat{A}(x, x)^T = ((A_1 - A_{-1})x, (A_1 - A_{-1})x)^T = \lambda(x, x)^T.$$

As the same way we can get the eigenvalues of $A_s$ is the eigenvalues of $\widehat{A}$. ∎

**Conjecture 11.4** (Bilu and Linial)**.** *Every d-regular graph has a signing in which all of the new eigenvalues (the eigenvalues of* $A_s$ *) have absolute value at most* $2\sqrt{d-1}$.

**Remark 11.5.** *This conjecture holds for bipartite graph, open for non-bipartite graph. Next we will prove this conjecture for bipartite graph.*

## 11.2 Matching Polynomial

**Definition 11.6.** *Let* $m_i$ *denote the number of matching in G consisting of i edges (*$m_0 = 0$*), the matching polynomial of G is*

$$u_G(x) = \sum_{i \geq 0} x^{n-2i}(-1)^i m_i.$$

**Theorem 11.7.**

$$\mathbb{E}_{s \in \pm 1^m}[f_s(x)] = u_G(x),$$

*where* $f_s(x)$ *is the characteristic polynomial of* $A_s$.

**Theorem 11.8.** *Let G be a graph with maximum degree at most d. then the roots of* $u_G(x)$ *are bounded in absolute value by* $2\sqrt{d-1}$*, moreover* $u_G(x)$ *is real rooted.*

*Proof of Theorem 11.7.*

$$\mathbb{E}[f_s(x)] = \mathbb{E}[det(xI - A_s)] = \mathbb{E}[\sum_{\pi \in S_m} sgn(\pi) x^{|\{a:\pi(a)=a\}|} \prod_{a:\pi(a) \neq a} -S(a, \pi(a))].$$

Consider the part

$$\prod_{a:\pi(a) \neq a} S(a, \pi(a)).$$

The only permutations that contribute to this sum are those for which $\pi(\pi(a)) = a$ for all $a$. Let $|\{a : \pi(a) = a\}| = m - 2i$. Every matching with size $i$ contribute to 1. So we have:

$$\mathbb{E}[\sum_{\pi\in S_m} sgn(\pi)x^{|\{a:\pi(a)=a\}|} \prod_{a:\pi(a)\neq a} -S(a,\pi(a))] = \sum_{i=0}^{\lfloor \frac{m}{2} \rfloor} sgn(\pi)x^{m-2i}m_i$$

Since the permutations just have the form

$$(a_1, a_1^{'})(a_2, a_2^{'}) \cdots (a_i, a_i^{'}),$$

then we have $sgn(\pi) = (-1)^i$. We have already finished the proof. ∎

**Lemma 11.9.** *Let $G$ be a tree and $A$ be its adjacent matrix. Then*

$$u_G(x) = f_x(A).$$

*Proof.*

$$f_x(A) = det(xI - A) = E[\sum_{\pi\in S_m} sgn(\pi)x^{|\{a:\pi(a)=a\}|} \prod_{a:\pi(a)\neq a} -A(a,\pi(a))].$$

∎

Since $G$ contains no cycles, so it contributes if and only if $\pi(\pi(a)) = a$ for all $a$, same as the proof of Theorem 11.7, we finishes the proof.

**Lemma 11.10.** *Let $G$ and $H$ be graphs on different vertex sets. Then*

$$u_{G\cup H}(x) = u_G(x)u_H(x).$$

*Proof.* Exercise. ∎

**Lemma 11.11.** $u_G(x) = xu_{(G-a)}(x) - \sum_{b\sim a} u_{(G-a-b)}(x)$.

*Proof.* Divide the matching in $G$ into two classes that involve $a$ or not. ∎

**Definition 11.12.** *Given a graph $G$ and a vertex $u$, the path tree $P(G, u)$ contains one vertex for every simple walk in $G$, beginning at $u$. Two vertices in $P(G, u)$ are adjacent if the simple walk corresponding to one is a continuation of the other.*

**Lemma 11.13.** *$G$ is a graph with maximum degree at most $d$. Then the root of $u_{P(G,u)}(x)$ is absolute bounded by $2\sqrt{d-1}$ for all $u \in V(G)$.*

To prove Theorem 11.8, we only need to prove the following lemma.

**Theorem 11.14.** *For every vertex $a$ of $G$, the polynomial $u_x[G]$ divides the polynomial $u_x[T_a(G)]$.*

We proved $u_x[T_a(G)] = f_x[T_a(G)]$, and the maximum degree of $T_a(G)$ is at most $d$, maximum root of $f_x[T_a(G)]$ at most $2\sqrt{d-1}$.

For $a$ a vertex of $G = (V, E)$, we write $G - a$ for the graph $G(V - \{a\})$. This notation will prove very useful when reasoning about matching polynomials. Fix a vertex $a$ of $G$, and divide the matchings in $G$ into two classes: those that involve vertex $a$ and those that do not. The number of matchings of size $k$ that do not involve $a$ is $m_k(G - a)$. On the other hand, those that do involve $a$ connect $a$ to one of its neighbors. To count these, we enumerate the neighbors $b$ of $a$. A matching of size $k$ that includes edge $(a, b)$ can be written as the union of $(a, b)$ and a matching of size $k-1$ in $G-a-b$. So, the number of matchings that involve $a$ is

$$\sum_{b \sim a} m_{k-1}(G-a-b).$$

This gives a recurrence for the number of matchings of size $k$ in $G$

$$m_k(G) = m_k(G-a) + \sum_{b \sim a} m_{k-1}(G-a-b).$$

To turn this into a recurrence for $\mu_x[G]$, write

$$x^{n-2k}(-1)^k m_k(G) = x \cdot x^{(n-1)-2k}(-1)^k m_k(G-a) - x^{(n-2)-2(k-1)}(-1)^{k-1} m_{k-1}(G-a-b).$$

This establishes the following formula.

**Lemma 11.15.**

$$\mu_x[G] = x\mu_x[G-a] - \sum_{b \sim a} \mu_x[G-a-b].$$

To prove theorem 11.14, we need the following lemma

**Lemma 11.16.** *For every graph $G$ and vertex $a$ of $G$,*

$$\frac{\mu_x[G-a]}{\mu_x[G]} = \frac{\mu_x\left[T_a(G)-a\right]}{\mu_x\left[T_a(G)\right]}.$$

*Proof.* If $G$ is a tree, then the left and right sides are identical, and so the inequality holds. As the only graphs on less than 3 vertices are trees, the theorem holds for all graphs on at most 2 vertices. We will now prove it by induction on the number of vertices. We may use Lemma 11.15 to expand the reciprocal of the left-hand side

$$\frac{\mu_x[G]}{\mu_x[G-a]} = \frac{x\mu_x[G-a] - \sum_{b \sim a} \mu_x[G-a-b]}{\mu_x[G-a]} = x - \sum_{b \sim a} \frac{\mu_x[G-a-b]}{\mu_x[G-a]}.$$

By applying the inductive hypothesis to $G-a$, we see that this equals

$$x - \sum_{b \sim a} \frac{\mu_x\left[T_b(G-a)-b\right]}{\mu_x\left[T_b(G-a)\right]}. \tag{11.4}$$

To simplify this expression, we examine these graphs carefully. By the observation we made before the proof,

$$T_b(G-a) - b = \bigcup_{c \sim b, c \neq a} T_c(G-a-b).$$

Similarly,

$$T_a(G) - a = \bigcup_{c \sim a} T_c(G-a),$$

which implies

$$\mu_x\left[T_a(G)-a\right] = \prod_{c \sim a} \mu_x\left[T_c(G-a)\right].$$

Let $ab$ be the vertex in $T_a(G)$ corresponding to the path from $a$ to $b$. We also have

$$\begin{aligned} T_a(G) - a - ab &= \left( \bigcup_{c \sim a, c \neq b} T_c(G-a) \right) \cup \left( \bigcup_{c \sim b, c \neq a} T_c(G-a-b) \right) \\ &= \left( \bigcup_{c \sim a, c \neq b} T_c(G-a) \right) \cup \left( T_b(G-a) - b \right), \end{aligned}$$

which implies

$$\mu_x \left[ T_a(G) - a - ab \right] = \left( \prod_{c \sim a, c \neq b} \mu_x \left[ T_c(G-a) \right] \right) \mu_x \left[ T_b(G-a) - b \right].$$

Thus,

$$\begin{aligned} \frac{\mu_x \left[ T_a(G) - a - ab \right]}{\mu_x \left[ T_a(G) - a \right]} &= \frac{\left( \prod_{c \sim a, c \neq b} \mu_x \left[ T_c(G-a) \right] \right) \mu_x \left[ T_b(G-a) - b \right]}{\prod_{c \sim a} \mu_x \left[ T_c(G-a) \right]} \\ &= \frac{\mu_x \left[ T_b(G-a) - b \right]}{\mu_x \left[ T_b(G-a) \right]}. \end{aligned}$$

Plugging this in to (11.4), we obtain

$$\begin{aligned} \frac{\mu_x[G]}{\mu_x[G-a]} &= x - \sum_{b \sim a} \frac{\mu_x \left[ T_a(G) - a - ab \right]}{\mu_x \left[ T_a(G) - a \right]} \\ &= \frac{x \mu_x \left[ T_a(G) - a \right] - \sum_{b \sim a} \mu_x \left[ T_a(G) - a - ab \right]}{\mu_x \left[ T_a(G) - a \right]}. \\ &= \frac{\mu_x \left[ T_a(G) \right]}{\mu_x \left[ T_a(G) - a \right]} \end{aligned}$$

Be obtain the equality claimed in the theorem by taking the reciprocals of both sides. ∎

Now we use it to prove Theorem 11.14.

*Proof.* We again prove this by induction on the number of vertices in $G$, using as our base case graphs with at most 2 vertices. We then know, by induction, that for $b \sim a$,

$$\mu_x[G-a] \text{ divides } \mu_x \left[ T_b(G-a) \right].$$

As

$$T_a(G) - a = \cup_{b \sim a} T_b(G-a)$$

$$\mu_x \left[ T_b(G-a) \right] \quad \text{divides} \quad \mu_x \left[ T_a(G) - a \right].$$

Thus,

$$\mu_x[G-a] \text{ divides } \mu_x \left[ T_a(G) - a \right],$$

and so

$$\frac{\mu_x \left[ T_a(G) - a \right]}{\mu_x[G-a]}$$

is a polynomial in $x$. To finish the proof, we apply Lemma 11.16, which implies

$$\mu_x \left[ T_a(G) \right] = \mu_x \left[ T_a(G) - a \right] \frac{\mu_x[G]}{\mu_x[G-a]} = \mu_x[G] \frac{\mu_x \left[ T_a(G) - a \right]}{\mu_x[G-a]}.$$

∎

### 11.3 Interlace Family

Now we introduce the main theorem of this class.

**Theorem 11.17** (main)**.** *Let $G$ be a graph with maximum degree at most $d$. Then there exists a signed adjacency matrix* $\mathbf{S}$ *with eigenvalues at most* $2\sqrt{d-1}$.

**Definition 11.18.** *If $p(x)$ is a real rooted polynomial of degree $n$ and $q(x)$ is a real rooted polynomial of degree $n-1$, then we say that $p$ and $q$ interlace if $p$ has roots $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and $q$ has roots $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1}$ that satisfy*

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \cdots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n.$$

*When $p$ and $q$ have the same degree, we also say that they interlace if their roots alternate. If*

$$p(x) = \prod_{i=1}^{n} (x - \lambda_i) \quad \text{and} \quad q(x) = \prod_{i=1}^{n} (x - \mu_i).$$

*we write $q \to p$ if $p$ and $q$ interlace and for every $i$ the $i$ th root of $p$ is at least as large as the $i$ th root of $q$. That is, if*

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \lambda_n \geq \mu_n.$$

For a polynomial $p$, let $\lambda_{\max}(p)$ denote its largest zero. When polynomials interlace, we can relate the largest zero of their sum to the largest zero of at least one of them.

**Lemma 11.19.** *Let $p_1(x), p_2(x)$ and $r(x)$ be polynomials so that $r(x) \to p_i(x)$. Then, $r(x) \to p_1(x) + p_2(x)$ and there is an $i \in \{1, 2\}$ for which*

$$\lambda_{\max}(p_i) \leq \lambda_{\max}(p_1 + p_2).$$

*Proof.* *Let $\mu_1$ be the largest zero of $r(x)$. As each polynomial $p_i(x)$ has a positive leading coefficient, each is eventually positive and so is their sum. As each has exactly one zero that is at least $\mu_1$ each is non-positive at $\mu_1$, and the same is also true of their sum. Let $\lambda$ be the largest zero of $p_1 + p_2$. We have established that $\lambda \geq \mu_1$.*

*If $p_i(\lambda) = 0$ for some $i$, then we are done. If not, there is an $i$ for which $p_i(\lambda) > 0$. As $p_i$ only has one zero larger than $\mu_1$, and it is eventually positive, the largest zero of $p_i$ must be less than $\lambda$.*

**Definition 11.20** (Laplacianized Polynomials)**.** *Instead of directly reasoning about the characteristic polynomials of signed adjacency matrices $S$, we will work with characteristic polynomials of $d\boldsymbol{I} - \boldsymbol{S}$. It suffices for us to prove that there exists an $S$ for which the largest eigenvalue of $d\boldsymbol{I} - \boldsymbol{S}$ is at most $d + 2\sqrt{d-1}$.*

*Fix an ordering on the $m$ edges of the graph, associate each $S$ with a vector $\sigma \in \{\pm 1\}^m$, and define*

$$p_\sigma(x) = \chi_x(d\boldsymbol{I} - \boldsymbol{S})$$

*The expected polynomial is the average of all these polynomials. We define two vectors for each edge in the graph. If the $i$ th edge is $(a, b)$, then we define*

$$v_{i,\sigma_1} = \delta_a - \sigma_i \delta_b$$

*For every $\sigma \in \{\pm 1\}^m$, we have*

$$\sum_{i=1}^{m} v_{i,\sigma_i} v_{i,\sigma_i}^T = d\boldsymbol{I} - \boldsymbol{S}$$

*where $S$ is the signed adjacency matrix corresponding to $\sigma$. So, for every $\sigma \in \{\pm 1\}^m$,*

$$p_\sigma(x) = \chi_x\left(\sum_{i=1}^{m} v_{i,\sigma_i} v_{i,\sigma_i}^T\right).$$

**Theorem 11.21.** *Let $\boldsymbol{A}$ be a symmetric matrix and let $\boldsymbol{w}_{i,s}$ be vectors for $1 \leq i \leq k$ and $s \in \{0,1\}$. Then the polynomial*

$$\sum_{\rho \in \{0,1\}^k} \chi_x\left(\boldsymbol{A} + \sum_{i=1}^{k} \boldsymbol{w}_{i,\rho_i} \boldsymbol{w}_{i,\rho_i}^T\right)$$

*is real rooted, and for each $s \in \{0,1\}$,*

$$\sum_{\rho \in \{0,1\}^k} \chi_x\left(\boldsymbol{A} + \sum_{i=1}^{k-1} w_{i,\rho_s} \boldsymbol{w}_{i,\rho_1}^T\right) \to \sum_{\rho \in \{0,1\}^k} \chi_x\left(\boldsymbol{A} + \sum_{i=1}^{k-1} \boldsymbol{w}_{i,\rho_i} \boldsymbol{w}_{i,\rho_i}^T + w_{k,s} \boldsymbol{w}_{k,s}^T\right).$$

**Lemma 11.22.** *Let $\boldsymbol{A}$ be a symmetric matrix and let $v$ be a vector. For a real number $t$ let*

$$p_t(x) = \chi_x\left(\boldsymbol{A} + t\boldsymbol{v}\boldsymbol{v}^T\right).$$

*Then, for $t > 0, p_0(x) \to p_t(x)$ and there is $n-1$ polynomial $q(x)$ so that for all $t$*

$$p_t(x) = \chi_x(\boldsymbol{A}) - tq(x).$$

*Proof.* The fact that $p_0(x) \to p_t(x)$ for $t > 0$ follows from the Courant-Fischer Theorem. We first establish the existence of $q(x)$ in the case that $v = \delta_1$. As the matrix $t\delta_1\delta_1^T$ is zeros everywhere except for the element $t$ in the upper left entry and the determinant is linear in each entry of the matrix, $\chi_x\left(\boldsymbol{A} + t\boldsymbol{\delta}_1\boldsymbol{\delta}_1^T\right) = \det\left(x\boldsymbol{I} - \boldsymbol{A} - t\boldsymbol{\delta}_1\boldsymbol{\delta}_1^T\right) = \det\left({}_1\boldsymbol{I} - \boldsymbol{A}\right) - t\det\left(x\boldsymbol{I}^{(1)} - \boldsymbol{A}^{(1)}\right) = \chi_x(\boldsymbol{A}) - t\chi_x\left(\boldsymbol{A}^{(1)}\right)$ where $\boldsymbol{A}^{(1)}$ is the submatrix of $\boldsymbol{A}$ obtained by removing its first row and column. The polynomial $q(x) = \chi_x\left(\boldsymbol{A}^{(1)}\right)$ has degree $n-1$ For arbitrary, $v$, let $Q$ be a rotation matrix for which $Qv = \delta_1$. As determinants, and thus characteristic polynomials, are unchanged by multiplication by rotation matrices,

$$\begin{aligned}\chi_x\left(\boldsymbol{A} + t\boldsymbol{v}v^T\right) &= \chi_x\left(\boldsymbol{Q}\left(\boldsymbol{A} + t\boldsymbol{v}\boldsymbol{v}^T\right)\boldsymbol{Q}^T\right)\\ &= \chi_x\left(\boldsymbol{Q}\boldsymbol{A}\boldsymbol{Q}^T + t\delta_1\delta_1^T)\right) = \chi_x\left(\boldsymbol{Q}\boldsymbol{A}\boldsymbol{Q}^T\right) - tq(x) = \chi_x(\boldsymbol{A}) - tq(x),\end{aligned}$$

for some $q(x)$ of degree $n-1$. ∎

**Lemma 11.23** (key)**.** *Let $p$ and $q$ be polynomials of degree $n$ and $n-1$, and let $p_t(x) = p(x) - tq(x)$. If $p_t$ is real rooted for all $t \in \mathbb{R}$, then $q$ interlaces $p$.*

*Proof.* Recall that the roots of a polynomial are continuous functions of its coefficients, and thus the roots of $p_t$ are continuous functions of $t$. We will use this fact to obtain a contradiction. For simplicity, [2] I just consider the case in which all of the roots of $p$ and $q$ are distinct. If they are not, one can prove this by dividing out their common divisors.

If $p$ and $q$ do not interlace, then $p$ must have two roots that do not have a root of $q$ between them. Let these roots of $p$ be $\lambda_{i+1}$ and $\lambda_i$. Assume, without loss of generality, that both $p$ and $q$ are positive between these roots. We now consider the behavior of $p_t$ for positive $t$.

As we have assumed that the roots of $p$ and $q$ are distinct, $q$ is positive at these roots, and so $p_t$ is negative at $\lambda_{i+1}$ and $\lambda_i$. If $t$ is very small, then $p_t$ will be close to $p$ in value, and so there must be some small $t_0$ for which $p_{t_0}(x) > 0$ for some $\lambda_{i+1} < x < \lambda_i$. This means that $p_{t_0}$ must have two roots between $\lambda_{i+1}$ and $\lambda_i$.

As $q$ is positive on the entire closed interval $[\lambda_{i+1}, \lambda_i]$, when $t$ is large $p_t$ will be negative on this entire interval, and thus have no roots inside. As we vary $t$ between $t_0$ and infinity, the two roots at $t_0$ must vary continuously and cannot cross $\lambda_{i+1}$ or $\lambda_i$. This means that they must become complex, contradicting our assumption that $p_t$ is always real rooted. ∎

**Lemma 11.24.** *Let $p$ and $q$ be polynomials of degree $n$ and $n-1$ that interlace and have positive leading coefficients. For every $t > 0$, define $p_t(x) = p(x) - tq(x)$. Then, $p_t(x)$ is real rooted and*

$$p(x) \to p_t(x).$$

*Proof.* For simplicity, consider the case in which all of the roots of $p$ and $q$ are distinct. One can prove the general case by dividing out the common repeated roots.

To see that the largest root of $p_t$ is larger than $\lambda_1$, note that $q(x)$ is positive for all $x > \mu_1$, and $\lambda_1 > \mu_1$. So, $p_t(\lambda_1) = p(\lambda_1) - tq(\lambda_1) < 0$. As $p_t$ is monic, it is eventually positive and it must have a root larger than $\lambda_1$.

We will now show that for every $i \geq 1, p_t$ has a root between $\lambda_{i+1}$ and $\lambda_i$. As this gives us $d-1$ more roots, it accounts for all $d$ roots of $p_t$. For $i$ odd, we know that $q(\lambda_i) > 0$ and $q(\lambda_{i+1}) < 0$. As $p$ is zero at both of these points, $p_t(\lambda_i) > 0$ and $p_t(\lambda_{i+1}) < 0$, which means that $p_t$ has a root between $\lambda_i$ and $\lambda_{i+1}$. The case of even $i$ is similar. ∎

To prove Theorem 11.17, it is sufficient for us to prove that there exists an $S$ for which the largest eigenvalue of $dI - S$ is at most $d + 2\sqrt{d-1}$.

Let $m$ be the number of edges in $G$, associate each $S$ with a $\sigma \in \{\pm 1\}^m$ with $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_m)$. Let $P_\sigma(x) = f_x(dI - S) = \det((x-d)I + S)$. Note that $\mathbb{E}[\det(xI - S)] = u_G(x)$, $\mathbb{E}[\det((x-d)I + S)] = u_G(x-d)$. And we have proved the following lemma

**Lemma 11.25.** *Let $G$ be a graph with maximum degree at most $d$, then the largest root of $u_G(x)$ is at most $2\sqrt{d-1}$.*

Let $\lambda_{max}(p)$ be the largest root of $p$. By Lemma 11.25, we have $\lambda_{max}(u_G(x-d)) \leq d + 2\sqrt{d-1}$. Let $v_{i\sigma_i} = \delta_a - \sigma_i \delta_b$, then $dI - S = \sum\limits_{i=1}^m v_{i\sigma_i} v_{i\sigma_i}^T$, $P_\sigma(x) = f_x\left(\sum\limits_{i=1}^m v_{i\sigma_i} v_{i\sigma_i}^T\right)$.

*Proof of the main theorem using Theorem 11.21.* Let $S^* = \left(S_1^*, \ldots, S_{m-k}^*\right) \in \{-1, 1\}^{m-k}$,

$$\phi_{S^*} = \sum_{\rho \in \{-1,1\}^k} f_x\left(\sum_{i=k+1}^m v_{iS^*_{m-i+1}} v^T_{iS^*_{m-i+1}} + \sum_{i=1}^k v_{i\rho_i} v^T_{i\rho_i}\right).$$

By Theorem 11.21, we have

$$\sum_{\rho\in\{-1,1\}^{k-1}} f_x\left(\sum_{i=k+1}^{m} v_{iS^*_{m-i+1}} v^T_{iS^*_{m-i+1}} + \sum_{i=1}^{k-1} v_{i\rho_i} v^T_{i\rho_i}\right) \to \phi_{(S^*,-1)}, \text{ and } \phi_{(S^*,1)}.$$

By Lemma 11.19, there exists an $i \in \{-1,1\}$ such that $\lambda_{max}\left(\phi_{(S^*,i)}\right) \le \lambda_{max}\left(\phi_{S^*}\right)$.

Let $\phi_\rho = \sum\limits_{\rho\in\{-1,1\}^m} f_x\left(\sum\limits_{i=1}^{m} v_{i\rho_i} v^T_{i\rho_i}\right)$, then $\phi_\rho = u_G(x-d)2^m$. By the analysis above, there is a $\sigma \in \{-1,1\}^m$ such that $\lambda_{max}(\phi_\sigma) \le \lambda_{max}(\phi_\rho) \le d + 2\sqrt{d-1}$. Let $S$ be the signed adjacency matrix corresponding to $\sigma$, we have $\lambda_{max}(f_x(dI - S)) = \lambda_{max}(\phi_\sigma) \le d + 2\sqrt{d-1}$. ∎

Next, we will prove Theorem 11.21. Recall that we have proved the following result.

**Lemma 11.26.** *Let $p$ and $q$ be polynomials with positive leading coefficient of degree $n$ and $n-1$ respectively, and let $p_t(x) = p(x) - tq(x)$.*

*(1) If $p_t$ is real rooted for all $t \in \mathbb{R}$, then $p$ interlaces $q$.*

*(2) If $p$ interlaces $q$, then for every $t > 0$ (resp. $t < 0$), $p(x) \to p_t(x)$ (resp. $p_t(x) \to p(x)$) and $p_t(x)$ is real rooted.*

**Lemma 11.27.** *Let $A$ be a symmetric matrix and let $v$ be a vector. For a real number $t$, let $p_t(x) = f_x\left(A + tvv^T\right)$. Then for $t > 0$, $p_0(x) \to p_t(x)$ and there is a degree $n-1$ polynomial $q(x)$ with positive leading coefficient so that for all $t$, $p_t(x) = f_x(A) - tq(x)$.*

*Proof of Theorem 11.21.* We prove it by induction on $k$. For $k = 1$, by Lemma 11.27, the statement is true.

Assuming that we have proven it for $k-1$.

For $k$, let $u$ be any vector and let $t \in \mathbb{R}$. Define

$$p_t(x,u) = \sum_{\rho\in\{-1,1\}^{k-1}} f_x\left(A + \sum_{i=1}^{k-1} w_{i\rho_i} w^T_{i\rho_i} + tuu^T\right).$$

By Lemma 11.27, we can express this polynomial in the form $p_t(x,u) = p_0(x) - tq(x,u)$, where $q(x,u)$ has positive leading coefficient and degree $n-1$. Let $A' = A + tuu^T$, by induction, $p_t(x,u)$ is real rooted for all $t$. By Lemma 11.26, we have $q(x,u)$ interlaces $p_0(x)$ and $p_0(x) \to p_t(x,u)$ for $t > 0$.

So $p_0(x) \to p_1(x, w_{k,1})$ and $p_0(x) \to p_1(x, w_{k,-1})$. By Lemma 11.19, $p_0(x) \to p_1(x, w_{k,1}) + p_1(x, w_{k,-1})$ is real rooted. ∎

## 12 Expander Graphs

Recall the Alon-Boppana bound, it tells us that the asymptotically best expanders are Ramanujan graphs. In this section, $G$ is a $d$-regular graph and

- $d = \mu_1 \ge \mu_2 \ge \cdots \ge \mu_n \ge -d$ are the eigenvalues of the adjacency matrix $A_G$.
- $0 = \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$ are the eigenvalues of the Laplacian matrix $L_G$.

## 12.1 Expanders as Approximations of the Complete Graph

**Definition 12.1.** *A d-regular graph is called an $\varepsilon$-expander if* $\max_{i\geq 2}|\mu_i| \leq \varepsilon d$. ($\max_{i\geq 2}|\mu_i| = \max\{\mu_2, |\mu_n|\}$).

**Definition 12.2.** *The norm of a matrix $M$ is* $\|M\| = \max_{\boldsymbol{x}\neq 0} \frac{\|M\boldsymbol{x}\|}{\|\boldsymbol{x}\|}$.

**Proposition 12.3.** *If $M$ is symmetric, then* $\|M\| = \max_i |\mu_i|$.

*Proof.* Exercise. ∎

Recall that we denote $H \preccurlyeq G$ if $\forall \boldsymbol{x}$, $\boldsymbol{x}^T L_H \boldsymbol{x} \leq \boldsymbol{x}^T L_G \boldsymbol{x}$. We view $c \cdot G$ as the weighted graph for $c \in \mathbb{R}$.

**Definition 12.4.** *We say $G$ is an $\varepsilon$-approximation of $H$ if* $(1-\varepsilon)H \preccurlyeq G \preccurlyeq (1+\varepsilon)H$.

**Remark 12.5.** *In Definition 12.4, $G$ is not necessarily regular graph.*

**Proposition 12.6.** *If $G$ is a d-regular $\varepsilon$-expander and $H = \frac{d}{n}K_n$, then $G$ is an $\varepsilon$-approximation of $H$ and* $\|L_G - L_H\| \leq \varepsilon d$.

*Proof.*

$$\begin{aligned}
G \text{ is an } \varepsilon\text{-expander } &\Leftrightarrow \max_{i\geq 2}|\mu_i| \leq \varepsilon d \\
&\Leftrightarrow \max_{i\geq 2}|d - \lambda_i| \leq \varepsilon d \\
&\Leftrightarrow \max_{i\geq 2}\left|\frac{d}{n}\lambda_i\left(L_{K_n}\right) - \lambda_i\left(L_G\right)\right| \leq \varepsilon d \\
&\overset{\lambda_1=0}{\Leftrightarrow} \max_{i}\left|\frac{d}{n}\lambda_i\left(L_{K_n}\right) - \lambda_i\left(L_G\right)\right| \leq \varepsilon d
\end{aligned}$$

Since the eigenvalues of $\frac{d}{n}L_{K_n}$ are $0, d, \ldots, d$, by the knowledge from linear algebra, the eigenvalues of $L_{\frac{d}{n}K_n} - L_G$ are $\frac{d}{n}\lambda_i\left(L_{K_n}\right) - \lambda_i\left(L_G\right)$. So by Proposition 12.3, we have

$$\begin{aligned}
G \text{ is an } \varepsilon\text{-expander } &\Leftrightarrow \max_{i}\left|\frac{d}{n}\lambda_i\left(L_{K_n}\right) - \lambda_i\left(L_G\right)\right| \leq \varepsilon d \\
&\Leftrightarrow \|L_G - L_H\| \leq \varepsilon d
\end{aligned}$$

It suffices to see $(1-\varepsilon)L_H \leq L_G \leq (1+\varepsilon)L_H$ or equivalently $-\varepsilon L_H \leq L_G - L_H \leq \varepsilon L_H$, this follows from $\|L_G - L_H\| \leq \varepsilon d$. ∎

**Remark 12.7.** *Proposition 12.6 tells us $\varepsilon$-expander is stronger than $\varepsilon$-approximation of* $\frac{d}{n}K_n$.

## 12.2 Quasi-random Properties of Expanders

**Definition 12.8.** *For a graph $G = (V, E)$ and $S, T \subseteq V$, define* $\overrightarrow{E}(S,T) = \{(u,v) : u \in S, v \in T \text{ and } uv \in E(G)\}$.

Note that $\overrightarrow{E}(S,T)$ consists of ordered pairs of adjacent vertices. When $S \cap T = \emptyset$, $\overrightarrow{E}(S,T) = e(S,T)$, when $S = T$, $\overrightarrow{E}(S,S) = 2e(S)$.

**Theorem 12.9** (Alon-Chung)**.** *Let $G$ be a $d$-regular $\varepsilon$-approximation of $\frac{d}{n}K_n$. Then for any $S, T \subseteq V(G)$ with $|S| = \alpha n$ and $|T| = \beta n$, we have $\left|\left|\overrightarrow{E}(S,T)\right| - \alpha\beta dn\right| \leq \varepsilon dn\sqrt{(\alpha-\alpha^2)(\beta-\beta^2)}$.*

*Proof.* Let $\boldsymbol{\chi}_S$ be the characteristic vector of the subset $S$, $\boldsymbol{\chi}_T$ be the characteristic vector of the subset $T$. Then

$$\boldsymbol{\chi}_S^T L_G \boldsymbol{\chi}_T = \sum_{i,j} \boldsymbol{\chi}_S(i) L_G(i,j) \boldsymbol{\chi}_T(j) = d|S \cap T| - \left|\overrightarrow{E}(S,T)\right|.$$

Let $H = \frac{d}{n}K_n$, we have

$$\boldsymbol{\chi}_S^T L_H \boldsymbol{\chi}_T = \boldsymbol{\chi}_S^T \left(dI - \frac{d}{n}J\right) \boldsymbol{\chi}_T = d|S \cap T| - \frac{d}{n}|S||T| = d|S \cap T| - \alpha\beta dn.$$

Then it is enough to show

$$\left|\boldsymbol{\chi}_S^T L_G \boldsymbol{\chi}_T - \boldsymbol{\chi}_S^T L_H \boldsymbol{\chi}_T\right| = \left|\left|\overrightarrow{E}(S,T)\right| - \alpha\beta dn\right| \leq \varepsilon dn\sqrt{(\alpha-\alpha^2)(\beta-\beta^2)}.$$

Notice that

$$\begin{aligned}\left|\boldsymbol{\chi}_S^T L_G \boldsymbol{\chi}_T - \boldsymbol{\chi}_S^T L_H \boldsymbol{\chi}_T\right| &= \left|\boldsymbol{\chi}_S^T \left(L_G - L_H\right) \boldsymbol{\chi}_T\right| \\ &\leq \left\|\boldsymbol{\chi}_S\right\| \left\|L_G - L_H\right\| \left\|\boldsymbol{\chi}_T\right\| \\ &\leq \varepsilon d \left\|\boldsymbol{\chi}_S\right\| \left\|\boldsymbol{\chi}_T\right\|.\end{aligned}$$

Also note that $\forall \boldsymbol{x}$ and $\forall c \in \mathbb{R}$, $L_H\boldsymbol{x} = L_H(\boldsymbol{x} + c\mathbf{1})$, $L_G\boldsymbol{x} = L_G(\boldsymbol{x} + c\mathbf{1})$. Instead of using $\boldsymbol{\chi}_S$ and $\boldsymbol{\chi}_T$, we can use $\boldsymbol{y}_S = \boldsymbol{\chi}_S - \alpha\mathbf{1}$ and $\boldsymbol{y}_T = \boldsymbol{\chi}_T - \beta\mathbf{1}$. So

$$\begin{aligned}\left|\left|\overrightarrow{E}(S,T)\right| - \alpha\beta dn\right| &= \left|\boldsymbol{\chi}_S^T \left(L_G - L_H\right) \boldsymbol{\chi}_T\right| \\ &= \left|\boldsymbol{y}_S^T \left(L_G - L_H\right) \boldsymbol{y}_T\right| \\ &\leq \varepsilon d \left\|\boldsymbol{y}_S\right\| \left\|\boldsymbol{y}_T\right\| \\ &= \varepsilon dn\sqrt{(\alpha-\alpha^2)(\beta-\beta^2)}.\end{aligned}$$

∎

**Remark 12.10.** *If $S$ and $T$ are disjoint, then the same proof will also work out for irregular weighted graph $G$ if we replace $\left|\overrightarrow{E}(S,T)\right|$ by $\omega(S,T) = \sum_{\substack{(u,v)\in E(G)\\ u\in S, v\in T}} \omega(u,v)$.*

**Definition 12.11.** *For $S \subseteq V(G)$, let $\tilde{N}(S)$ denote the set of vertices that are neighhours of vertices in $S$, i.e., $\tilde{N}(S) = \{x \in V(G) : x \sim y, y \in S\}$. Note that $\tilde{N}(S)$ may contain vertices in $S$.*

**Theorem 12.12** (Tanner's theorem)**.** *Let $G$ be a $d$-regular graph on $n$ vertices and $\epsilon$ -approximation of $\frac{d}{n}K_n$. Then, for any $S \subseteq V$ with $|S| = \alpha n$,*

$$|\tilde{N}(S)| \geq \frac{|S|}{\epsilon^2(1-\alpha)+\alpha}.$$

*Proof.* Let $R = \tilde{N}(S)$ and let $T = V - R$. Then, $e(S,T) = 0$ since there are no edges between $S$ and $T$. Let $|T| = \beta n$ and $|R| = \gamma n$, where $\gamma = 1 - \beta$. By Theorem 12.9(Alon-Chung), we have

$$\begin{aligned}
& \alpha\beta dn \leq \epsilon dn\sqrt{(\alpha-\alpha^2)(\beta-\beta^2)} \\
\Leftrightarrow \quad & \alpha^2\beta^2 \leq \epsilon^2\left(\alpha-\alpha^2\right)\left(\beta-\beta^2\right) \\
\Leftrightarrow \quad & \alpha\beta \leq \epsilon^2(1-\alpha)(1-\beta) \\
\Leftrightarrow \quad & \frac{1-\gamma}{\gamma} = \frac{\beta}{1-\beta} \leq \frac{\epsilon^2(1-\alpha)}{\alpha} \\
\Leftrightarrow \quad & \frac{1}{\gamma} \leq \frac{\epsilon^2(1-\alpha)+\alpha}{\alpha} \\
\Leftrightarrow \quad & \gamma \geq \frac{\alpha}{\epsilon^2(1-\alpha)+\alpha} \\
\Leftrightarrow \quad & \gamma n \geq \frac{\alpha n}{\epsilon^2(1-\alpha)+\alpha} \\
\Leftrightarrow \quad & |\tilde{N}(S)| \geq \frac{|S|}{\epsilon^2(1-\alpha)+\alpha}.
\end{aligned}$$

∎

**Exercise 8.** *Prove that if $G$ is $\epsilon$-approximation of $H = \frac{d}{n}K_n$ with average degree $d$, then the same conclusion holds.*

**Fact 12.13.** *If we take $S$ as $u$ in Theorem 12.12(Tanner's theorem), we have*

$$d = |\tilde{N}(u)| \geq \frac{|S|}{\epsilon^2(1-\alpha)+\alpha} = \frac{1}{\epsilon^2(1-\frac{1}{n})+\frac{1}{n}},$$

*which implies that*

$$\epsilon \geq \frac{1}{\sqrt{d+\frac{d^2}{n}}} \approx \frac{1}{\sqrt{d}}.$$

**Exercise 9.** *Prove that for fixed $d$, if $G$ is an $n$-vertex and $d$- regular graph, then there exist two edges $xy$ and $zw$ whose distance is at least $\Omega(\log n)$.*

**Theorem 12.14** (A.Nilli, a stronger version of Alon-Boppana)**.** *Let $G$ be a $d$-regular graph containing two edges $(u_0, u_1)$ and $(v_0, v_1)$ that are at distance at least $2k+2$. Then*

$$\lambda_2 \leq d - 2\sqrt{d-1} + \frac{2\sqrt{d-1}-1}{k+1}.$$

*Proof.* Recall that $\lambda_2 = \min\limits_{\vec{x}\perp\vec{1}} \frac{\vec{x}^T L_G \vec{x}}{\vec{x}^T\vec{x}}$. To prove the upper bound of $\lambda_2$, we need to find a "good" $\vec{x}$ that is perpendicular to $\vec{1}$. We let $U_0 = u_0, u_1$ and $V_0 = v_0, v_1$. Let $U_i$ be the set of vertices at distance $i$ from $u_0$ and let $V_j$ be the set of vertices at distance $j$ from $v_0$. We know that $U_0, U_1, ..., U_k, V_0, V_1, ..., V_k$ are vertex-disjoint and $e(U_k, V_k) = 0$. We are going to find a test vector. Our test vector for $\lambda_2$ will be given by Our test vector for $\lambda_2$ will he given by

$$\vec{x}(a) = \begin{cases} (d-1)^{-i/2} & \text{if } a \in U_i \text{ and } i \leq k, \\ -\beta(d-1)^{-j/2} & \text{if } a \in V_j \text{ and } j \leq k, \\ 0 & \text{otherwise.} \end{cases}$$

Hence $\beta$ is chosen so that $\vec{x}$ is orthogonal to $\vec{1}$, i.e.,$\vec{x}^T\vec{1} = 0$. Then

$$\lambda_2 \le \frac{\vec{x}^T L_G \vec{x}}{\vec{x}^T\vec{x}} \le \frac{X_0 + \beta^2 Y_0}{X_1 + \beta^2 Y_1} \le \max\left\{\frac{X_0}{X_1}, \frac{Y_0}{Y_1}\right\},$$

where

$$X_0 = \sum_{i=0}^{k-1} |U_i|(d-1)\left(\frac{1-1/\sqrt{d-1}}{(d-1)^{i/2}}\right)^2 + |U_k|(d-1)^{-k+1}, \text{ and } X_1 = \sum_{i=0}^{k} |U_i|(d-1)^{-i},$$

and

$$Y_0 = \sum_{j=0}^{k-1} |V_j|(d-1)\left(\frac{1-1/\sqrt{d-1}}{(d-1)^{j/2}}\right)^2 + |V_k|(d-1)^{-k+1}, \text{ and } Y_1 = \sum_{j=0}^{k} |V_j|(d-1)^{-j}.$$

Next we estimate $\frac{X_0}{X_1}$ as follows.

$$\begin{aligned}
X_0 &= \sum_{i=0}^{k-1} |U_i|(d-1)\left(\frac{1-1/\sqrt{d-1}}{(d-1)^{i/2}}\right)^2 + |U_k|(d-1)^{-k+1} \\
&= \sum_{i=0}^{k-1} \frac{|U_i|}{(d-1)^i}(\sqrt{d-1}-1)^2 + |U_k|(d-1)^{-k+1} \\
&= \sum_{i=0}^{k} \frac{|U_i|}{(d-1)^i}(d-2\sqrt{d-1}) + \frac{|U_k|}{(d-1)^k}(2\sqrt{d-1}-1) \\
&= X_1(d-2\sqrt{d-1}) + \frac{|U_k|}{(d-1)^k}(2\sqrt{d-1}-1).
\end{aligned}$$

Since $|U_i|(d-1)^{k-i} \ge |U_k|$ implies that $\frac{|U_k|}{(d-1)^k} \ge \frac{|U_i|}{(d-1)^i}$, we have

$$\frac{|U_k|}{(d-1)^k} \le \frac{1}{k+1}\sum_{i=0}^{k} \frac{|U_i|}{(d-1)^i} = \frac{X_1}{k+1}.$$

Combining, we see that

$$\frac{X_0}{X_1} \le d - 2\sqrt{d-1} + \frac{2\sqrt{d-1}-1}{k+1}.$$

So is $\frac{Y_0}{Y_1}$. Therefore

$$\lambda_2 \le \max\left\{\frac{X_0}{X_1}, \frac{Y_0}{Y_1}\right\} \le d - 2\sqrt{d-1} + \frac{2\sqrt{d-1}-1}{k+1}.$$

∎

**Project 3.** Try to prove the following or give a counter example. Let $G$ be the same as that in Theorem 12.14(A.Nilli), then

$$\lambda_n \ge d - 2\sqrt{d-1} + \frac{2\sqrt{d-1}-1}{k+1}.$$

or close enough.

**Project 4.** Try to generalize Theorem 12.14(A.Nilli) to some graph $G$ which is $\epsilon$-approximation of $\frac{d}{n}K_n$ and has average degree of $d$.

# 13 The Second Proof of Cheeger's Inequality

**Definition 13.1.** *For a graph $G$. the* conductance *of a subset $S \subseteq V(G)$ is $\phi(S) = \frac{|\partial S|}{\min\{d(S), d(V \backslash S)\}}$, where $d(S) = \sum_{x \in S} d_G(S)$. Let $\phi(G) = \min_{S \subseteq V(G)} \phi(G)$ be the* conductance *of $G$.*

**Definition 13.2.** *Given a vector $\vec{f}$ and an integer $k$, let $\vec{f}\{k\}$ be the sum of the $k$ largest entries of $\vec{f}$. Also let $\vec{f}\{0\} = 0$. For a real number $x \in [0, n]$ define $\vec{f}\{x\}$ by making it piecewise linear between integers. That is $\vec{f}\{x\} = (\lceil x \rceil - x)\vec{f}\{\lfloor x \rfloor\} = (x - \lfloor x \rfloor)\vec{f}\{\lceil x \rceil\})$.*

**Lemma 13.3.** *Let $h(x)$ be a convex function, and let $z > y > 0$. Then,*

$$\frac{1}{2}(h(x-z) + h(x+z)) \leq \frac{1}{2}(h(x-y) + h(x+y)).$$

**Lemma 13.4.** *Let $f$ be a vector, let $k \in [0, n]$, and let $\alpha_1, ..., \alpha_n$ be numbers between 0 and 1 such that*

$$\sum_i \alpha_i = k.$$

*Then,*

$$\sum_i \alpha_i f(i) \leq f\{k\}.$$

*Proof.* Exercise. ∎

**Recall** The *lazy random walk* on a (weight)graph $G$ is given by $W = \frac{1}{2}(I + MD^{-1})$, where $M = (w(i,j))_{x \times n}$ and $w(i,j)$ is the weight on $(i,j)$.

**Definition 13.5.** *For $S \subseteq V(G)$ and $a \in V(G)$, let $\gamma(a, S)$ be the probability that a random walk( that is at vertex $a$) move to a vertex in $S$ in one step.*

**Definition 13.6.** *Let $\vec{f}(S) = \sum_{a \in S} \vec{f}(a)$, For any $k \in \mathbb{Z}$ there is a $S \subseteq V(G)$ of size $k$ such that $\vec{f}(S) = \vec{f}\{k\}$.*

**Lemma 13.7.** *Let $G$ be a $d$-regular graph, let $\vec{f}$ be a vector and $\vec{g} = W\vec{f}$. Then $\vec{g}\{x\} \leq \vec{f}\{x\}$ for any $x \in [0, n]$.*

*Proof.* It is suffices to prove it for integers $k$. There is a sub set $S$ of size $k$ with $\vec{g}(S) = \vec{g}\{k\}$. Then $\vec{g}(S) = \sum_{a \in V} \gamma(a, S)\vec{f}(a)$. As $G$ is regular, we have

$$\sum_{a \in V} \gamma(a, S) = |S| = k.$$

By lemma 13.4, we have

$$\vec{g}(S) = \sum_{a \in V} \gamma(a, S)\vec{f}(a) \leq \vec{f}\{k\}.$$

∎

Note that $G$ is $d$-regular,

$$\phi(S) = \frac{|\partial S|}{\min\{d(S), d(V \setminus S)\}}.$$

**Lemma 13.8.** *Let $S \subseteq V(G)$ be of size $k$. Then $\sum_{\notin S} \gamma(a,S) = \frac{\phi(S)}{2} \cdot \min\{k, n-k\}$.*

*Proof.* As $a \notin S$, $\gamma(a,S)$ equals half the fraction of the edge from $a$ to $S$, i.e. $\gamma(a,S) = \frac{1}{2}\sum_{b\in S}\frac{w(a,b)}{d}$, which implies that

$$\sum_{a\notin S}\gamma(a,S) = \frac{1}{2}\frac{|\partial S|}{d} = \frac{\phi(S) d \min\{k,n-k\}}{2d} = \frac{\phi(S)}{2}\cdot\min\{k,n-k\}.$$

∎

**Lemma 13.9** (Key lemma)**.** *Let $W$ be the transition matrix of the lazy random walk on a $d$-regular graph, and let $\vec{g} = W\vec{f}$ . For every set $S$ of size $k$ with conductance at least $\phi$,*

$$\vec{g}(S) \leq \frac{1}{2}(f\{k-\phi h\} + f\{k+\phi h\}),$$

*where $h = \min(k, n-k)$.*

*Proof.* Fix $S$ and let $\gamma(a) = \gamma(a,S)$. We prove this by rearranging the formula $\vec{g}(S) = \sum_{a\in V}\gamma(a)\vec{f}(a)$. Also $\sum_{a\in V} = k$.

Let $\alpha(a) = \begin{cases} \gamma(a) - \frac{1}{2} if a \in S, \\ 0 if a \notin S, \end{cases}$ and $\beta(a) = \begin{cases} \frac{1}{2} if a \in S, \\ \gamma(a) if a \notin S, \end{cases}$ which implies that $\alpha(a)+\beta(a) = \gamma(a)$. So $\vec{g}(s) = \sum_{a\in V}\alpha(a)\vec{f}(a) + \sum_{a\in V}\beta(a)\vec{f}(a)$. We have that $a \in S$ if and only if $\gamma(a) \geq \frac{1}{2}$, which implies that $o \leq \alpha(a) \leq 1/2$ for all $a \in V$. Similar, we have $0 \leq \beta(a) \leq 1/2$ for all $a \in V$.

Then we write $\sum_{a\in V}\alpha(a)\vec{f}(a) = \frac{1}{2}\sum_{a\in V}2\alpha(a)\vec{f}(a)$, and $\sum_{a\in V}\beta(a)\vec{f}(a) = \frac{1}{2}\sum_{a\in V}2\beta(a)\vec{f}(a)$, where all coefficient $2\alpha, 2\beta \in [0,1]$. Let $z = \sum_{a\notin S}\gamma(a)$. Then $\sum_{a\in V}(2\beta(a)) = 2(\frac{k}{2}+\sum_{a\notin S}\gamma(a)) = k+2z$, and $\sum_{a\in V}(2\aleph(a)) = 2\sum_{a\in V}2\gamma(a) - (k+2z) = k-2z$. By Lemma 13.8, we have

$$z \geq \frac{h}{2}\phi(S) \geq \frac{h\phi}{2}.$$

By Lemma 13.4, we have

$$\vec{g}(k) \leq \frac{1}{2}(f\{k-2z\} + f\{k+2z\} \leq \frac{1}{2}(f\{k-\phi h\} + f\{k+\phi h\}).$$

This finishes the proof. ∎

By the key lemma, if the conductance of $G$ is big, then $\vec{g}(S) = \vec{g}\{k\}$ is smaller for some $S \subset V(G)$. Hence we can estimate the converge speed of $\vec{f_t} = W^t\vec{f_0}$. We know the sequence converges to $\vec{\pi} = \frac{1}{n}\vec{1}$, so the non-increasing sequence of concave functions $\{\vec{f_t}\{x\}\}$ converges to a linear function $\vec{\pi}\{x\} = \frac{x}{n}$.

**Theorem 13.10** (Cheeger's inequality, for $d$-regular graphs)**.** *Let $G$ be a $d$*-regular graph *with the lazy random walk whose transition matrix is $W$, and let $\omega_2 = 1-\lambda$ be the second largest eigenvalue of $W$. Then there exists a subset $S \subset V(G)$ such that $\phi(S) \leq \sqrt{8\lambda}$.*

**Remark 13.11.** *We know that $\omega_2 = 1 - \frac{1}{2}\nu_2$, so $\lambda = \frac{1}{2}\nu_2$ and $\phi(S) \leq 2\sqrt{\nu_2}$, where $\nu_2$ is the second smallest eigenvalue of the normalized Laplacian matrix $N_G = I - \frac{1}{d}M$. By the way, our first proof gives $\phi(S) \leq \sqrt{2\nu_2}$.*

*Proof.* Let $\vec{f}$ be the eigenvector of $W$ corresponding to the eigenvalue $\omega_2$. That is, $W\vec{f} = \omega_2\vec{f}$. Also we have $W = \frac{1}{2}(I + \frac{1}{d}M) = I - \frac{1}{2d}L_G$, so $\vec{f}$ is an eigenvector of $L_G$ corresponding to the second smallest eigenvalue $\lambda_2$, hence $\vec{f} \perp \vec{1}$. That is, $\vec{f}\{n\} = 0$.

Let $\gamma = \max_k \frac{\vec{f}\{k\}}{\sqrt{\min\{k,n-k\}}}$, and let $k$ be the integer achieving $\gamma$. Let $S \subset V(G)$ be a $k$-subset of $V(G)$ such that $\vec{f}(S) = \vec{f}\{k\}$. Then we need to prove that this $S$ satisfies the problem.

Since $W\vec{f} = \omega_2\vec{f}$ and $\omega_2 \in [0,1]$, we have $W\vec{f}\{k\} = W\vec{f}(S)$. Without losing of generality, we may assume $h = \min\{k, n-k\} = k$, since otherwise the proof is similar. By key lemma we have: $(W\vec{f})(S) \le \frac{1}{2}(\vec{f}\{k-\phi k\} + \vec{f}\{k+\phi k\}) \le \frac{1}{2}(\gamma\sqrt{k-\phi k} + \gamma\sqrt{k+\phi k})$, where the second inequality comes from the definition of $\gamma$. Also we have: $(W\vec{f})(S) = \omega_2\vec{f}(S) = \omega_2\vec{f}\{k\} = (1-\lambda)\gamma\sqrt{k}$. Hence we have: $(1-\lambda)\gamma\sqrt{k} \le \frac{1}{2}(\gamma\sqrt{k-\phi k} + \gamma\sqrt{k+\phi k})$, that is, $1-\lambda \le \frac{1}{2}(\sqrt{1-\phi} + \sqrt{1+\phi}) \le 1 - \frac{1}{8}\phi^2$, where the last inequality comes from Taylor's expansion. This indicates that $\phi \le \sqrt{8\lambda}$. ∎

# 14 Iterative Solvers for Linear Equations

We consider the linear system $A\vec{x} = \vec{b}$, where $A$ is a symmetric positive-definite matrix with eigenvalues $0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$. Our goal is to estimate the solution $\vec{x} = A^{-1}\vec{b}$ fast.

## 14.1 First-order Iteration

We know that $A\vec{x} = \vec{b}$, so for some $\alpha \in \mathbb{R}$, $\alpha A\vec{x} = \alpha\vec{b}$, hence $\vec{x} = (I - \alpha A)\vec{x} + \alpha\vec{b}$. This leads us to the following iterative process:

$$\begin{cases} \vec{x}_t = (I - \alpha A)\vec{x}_{t-1} + \alpha\vec{b} \\ \vec{x}_0 = \vec{0} \end{cases} \tag{14.5}$$

So the problem is: how to determine a proper $\alpha$?

**Theorem 14.1.** *For the iterative process (2.1), $\{\vec{x}_t\}$ converges if $\|I - \alpha A\| < 1$, where the matrix norm is defined by:*

$$\|M\| = \max_{\vec{x} \ne \vec{0}} \frac{\|M\vec{x}\|}{\|\vec{x}\|}.$$

**Remark 14.2.** *This means if $\max\{|1-\alpha\lambda_1|, |1-\alpha\lambda_n|\} < 1$, then $\{\vec{x}_t\}$ converges.*

*Proof.* It is enough to prove that $\{\vec{x}_t - A^{-1}\vec{b}\} \to \vec{0}$ as $t$ approaches infinity. We know $\vec{x}_t - A^{-1}\vec{b} = (I - \alpha A)(\vec{x}_{t-1} - A^{-1}\vec{b})$, so we have $\vec{x}_t - A^{-1}\vec{b} = (I - \alpha A)^t(-A^{-1}\vec{b})$. By considering the norm we have: $\|\vec{x}_t - A^{-1}\vec{b}\| \le \|(I - \alpha A)\|^t \|(-A^{-1}\vec{b})\| \to 0$ as $t$ approaches infinity. That is what we want. ∎

**Definition 14.3.** *For a positive-definite symmetric matrix $A$ whose eigenvalues are $0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$, we define its* condition number *to be $\kappa(A) = \frac{\lambda_n}{\lambda_1}$.*

**Remark 14.4.** *1. We take $\alpha$ such that $|1-\alpha\lambda_1| = |1-\alpha\lambda_n|$, which means $\alpha = \frac{2}{\lambda_1+\lambda_n}$. Then we have $\|I - \alpha A\| = \frac{\lambda_n - \lambda_1}{\lambda_n + \lambda_1} = 1 - \frac{2}{\kappa(A)+1}$.*

*2. In case we want to find a close $\vec{x}_t$ so that $\|\vec{x}_t - \vec{x}\| \le \epsilon\|\vec{x}\|$, then the iterative process (2.1) will take $\frac{1}{2}(\kappa(A)+1)\log(\frac{1}{\epsilon})$ time, as $(1 - \frac{2}{\kappa(A)+1})^t \le \epsilon$ indicates $t \ge \frac{\log\epsilon}{\log(1-\frac{2}{\kappa(A)+1})} \ge \frac{1}{2}(\kappa(A)+1)\log(\frac{1}{\epsilon})$.*

## 14.2 Another Method

From the iterative process (2.1), we have

$$\vec{x}_t = (\sum_{i=0}^{t-1}(I-\alpha A)^i)\alpha\vec{b} = p_t(A)\vec{b},$$

where $p_t$ is a polynomial of degree $t-1$.

Also, since $\|I-\alpha A\| < 1$, we have

$$\alpha\sum_{i=0}^{\infty}(I-\alpha A) = \alpha(I-(I-\alpha A))^{-1} = A^{-1}.$$

That is, $\vec{x}_t$ can be viewed as a *cut off* of $\vec{x}$.

To optimize the first-order iteration, it suffices to find better polynomials $q_t$ than $p_t$. In general, we aim to find polynomials $q_t(x)$ such that $\|\vec{x}-q_t(A)\vec{b}\| \le \epsilon\|\vec{x}\|$.

**Remark 14.5.** *The first-order iteration process tells us that $q_t(x) = \sum_{i=0}^{t-1}(1-\alpha x)^i$ satisfies the requirement.*

As $\vec{b} = A\vec{x}$, we have $\|\vec{x}-q_t(A)A\vec{x}\| \le \epsilon\|\vec{x}\|$, which can be derived from $\|I-q_t(A)A\| \le \epsilon$. Since we know the eigenvalues of $A$ are $0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$, it suffices to find a proper $q_t(x)$ such that $\max_i |1-q_t(\lambda_i)\lambda_i| \le \epsilon$. Thus, our aim is to find proper $q_t(x)$ such that $1-xq^t(x)$ is small in the interval $[\lambda_1, \lambda_n]$.

**Theorem 14.6.** *For any $t \ge 1$ and $0 < \lambda_{min} \le \lambda_{max}$, there exists a polynomial $g_t(x)$ of degree $t$ such that for any $x \in [\lambda_{min}, \lambda_{max}]$, $|g_t(x)| \le \epsilon$, and $g_t(0) = 1$, for $\epsilon \le 2(1+\frac{2}{\sqrt{\kappa}})^{-t} \le 2e^{-\frac{2t}{\sqrt{\kappa}}}$, where $\kappa = \frac{\lambda_{max}}{\lambda_{min}}$.*

*Proof.* This proof is assigned as exercise.

Hint: Take $l(x) = \frac{\lambda_{min}+\lambda_{max}-2x}{\lambda_{max}-\lambda_{min}}$, and let $g_t(x) = \frac{T_t(l(x))}{T_t(l(0))}$, where $T_t(x)$ denotes the t-th Chebyshev polynomial. ∎

**Remark 14.7.** *This method needs about $\sqrt{\kappa(A)}\log(\frac{1}{\epsilon})$ time, better compared with the first-order iteration method, which takes about $\frac{1}{2}(\kappa(A)+1)\log(\frac{1}{\epsilon})$ time.*

## 14.3 Chebyshev Polynomials

In this section, we give the definition of the Chebyshev polynomials, and introduce some of its basic properties.

**Definition 14.8.** *The $t$-th* Chebyshev polynomial *$T_t(x)$ has degree $t$ and can be defined by the following iterative process:*

$$\begin{cases} T_0(x) = 1 \\ T_1(x) = x \\ T_t(x) = 2xT_{t-1}(x) - T_{t-2}(x). \end{cases} \tag{14.6}$$

Next we give some basic properties.

**Proposition 14.9.** *Let $T_t(x)$ be the t-th Chebyshev polynomial.*

*1. $T_t(\cos(\theta)) = \cos(t\theta)$ and $T_t(\cosh(\theta)) = \cosh(t\theta)$, where $\cos(\theta) = \frac{1}{2}(e^{i\theta}+e^{-i\theta})$ and $\cosh(\theta) = \frac{1}{2}(e^{\theta}+e^{-\theta})$;*

*2. For any $t \geq 0$ and $x \in [-1,1]$, $|T_t(x)| \leq 1$;*

*3. Let $\operatorname{acosh}(x) = \ln(x+\sqrt{x^2-1})$ for $x \geq 1$, which is the inverse function of $\cosh(x)$. Then we have: $T_t(x) = \frac{1}{2}(e^{\operatorname{acosh}(x)t} + e^{-\operatorname{acosh}(x)t})$;*

*4. For any $\gamma > 0$, $T_t(1+\gamma) \geq \frac{1}{2}(1+\sqrt{2\gamma})^t$.*

*Proof.* The proofs for these properties are all assigned as homework, except the second property, which is a trivial corollary of the first property. ∎

# 15 An Application of Borsuk-Ulam Theorem

Suppose that two thieves have stolen a necklace with stones of various kinds. The thieves do not know the values of the stones so that they want to divide the stones of each kind evenly. And they want to achieve this by as few cuts as possible.

We are given an opened necklace with $kn$ stones, where stones have $t$ different colors(i.e. $k$-types). Suppose there are $ka_i$ stones of color $i$, for $1 \leq i \leq t$. So $\sum_{i=1}^{t} a_i = n$.

**Definition 15.1.** *A $k$-splitting of the necklace is a partition of the necklace into $k$ part, each consisting of a finite number of non overlapping intervals of stones whose union captures precisely $a_i$ stones of color $i$ for all $1 \leq i \leq t$.*

**Definition 15.2.** *The size of the $k$-splitting is the number of cuts that form the intervals of splitting (which is one less that the total number of intervals).*

**Theorem 15.3** (Alon)**.** *Every opened necklace with $ka_i$ stones of color $i$ for all $1 \leq i \leq t$ has a $k$-splitting of size at most $(k-1)t$. And this number $(k-1)t$ is best possible.*

**Theorem 15.4.** *Let $\mu_1, ..., \mu_t$ be $t$ continues probability measures on the interval $[0,1]$. Then it is possible to cut the interval in $(k-1)t$ places and partition the $(k-1)t+1$ resulting intervals into $k$ families $F_1, ...F_k$ such that*

$$\mu_i(\cup F_j) = 1/k \text{ for each } 1 \leq i \leq t, 1 \leq j \leq k.$$

*And the number $(k-1)t$ is best possible.*

Both Theorem 15.3 and Theorem 15.4 will follow a continues version of Theorem 15.3, which now we introduce. Let $I = [0,1]$ be the unit interval.

**Definition 15.5.** *An interval $t$-coloring is a coloring of points of $I$ by $t$ colors, such that for each $1 \leq i \leq t$, the set of $L_i$ of points colored by $i$ is Lebesgue measure.*

**Definition 15.6.** *A $k$-splitting of size $r$ on $I$ is a sequence of numbers $0 = y_0 < y_1 < ... < y_r < y_{r+1} = 1$ and a partition of family of $r+1$ intervals $F = \{[y_i, y_{i+1}]; 0 \leq i \leq r\}$ into $k$ disjoint subfamilies $F_1, F_2, ..., F_k$ such that $\cup F_i = F$ and for each $1 \leq j \leq k$, the number of the interval in $F_j$ captures precisely $1/k$ of the total measure of each of $t$ colors (i.e. $||(\cup F_i) \cap L_i|| = \frac{1}{k}||L_i||$ for all $1 \leq i \leq t, 1 \leq j \leq k$.)*

**Theorem 15.7** (Alon)**.** *Every interval $t$-coloring has a $k$-splitting of size at most $(k-1)t$.*

*Proof of Theroem 15.3(Assuming Theroem 15.7).* We are given a necklace with $ka_i$ stones of color $i$, $1 \le i \le t$. First we want to convert it into an interval $t$-coloring by partition $I = [0, 1]$ into $kn$ equal segment ($n = \sum_{i=1}^{t} a_i$) and coloring the $j^{th}$ segment by the same color of the $j^{th}$ stone.

By Theorem 15.7, there is a $k$ splitting on $I$ into $k$ families of intervals, say $F_1, ..., F_k$, with at most $(k-1)t$ cuts, but these cuts need not occur at the endpoint of the $kn$ segments. We say a cut is bad if this cut does not occur at an endpoint of some segment. So it suffices to show that there is a choice of $F_1, ...F_k$ such that the number $r$ of bad cuts is 0. Let $r$ be the least number of bad cuts. Assume that $r \ge 1$. (Otherwise, we are done.) Then there is some $i$ with $1 \le i \le t$ such that there is a bad cut in the interior of some segment belonging to color $i$.

construct a multigraph $G$ on the set of vertices $F_1, F_2, ..., F_k$. where $F_j \sim F_\ell$ if and only if there is a bad cut in color $i$ between an interval belonging to $F_j$ and an interval belonging to $F_\ell$.
**Claim.** Any vertex $F_j$ in $G$ with positive edges has degree at least 2.

*Proof of Claim.* Since the measure of color $i$ captured by $F_j$ is an integer multiple of $1/kn$, the degree of $F_j$ in $G$ cannot be 1. ∎

Therefore, $G$ has a cycle. We can now slide all the cuts corresponds the edges of the cycles by the same amount, without changing the measure of each color captured by all $F_j$'s, until one of the cuts reaches the bounding of its small segment.

This will decrease $r$ by at least 1, a contradiction. This finishes the proof of Theorem 15.3. ∎

Next we want to prove Theorem 15.7, using the following three lemmas.

**Lemma 15.8.** *Theorem 15.7 holds for $k = 2$ (for every $t$).*

**Lemma 15.9.** *If Theorem 15.7 holds for $(t, k)$ and for $(t, \ell)$, then Theorem 15.7 holds for $(t, k\ell)$.*

*Proof.* Exercise. ∎

**Lemma 15.10.** *Theorem 15.7 holds for every odd prime $k$ (for every $t$).*

We prove Lemma 15.8 by the following Borsuk-Ulam Theorem.

**Theorem 15.11** (Borsuk-Ulam)**.** *Let $f : S^k \to \mathbb{R}^k$ be a continues function from the k-dimensional sphere $S^k$ (in $\mathbb{R}^{k+1}$) to $\mathbb{R}^k$ such that $f(\vec{x}) = f(-\vec{x})$ for all $\vec{x} \in S^k$. Then there exists a $\vec{x} \in S^k$ with $f(\vec{x}) = 0$.*

*Proof of Lemma 15.8.* We are given an interval $t$-coloring on $I = [0, 1]$ and $k = 2$. Define a function $f : S^t \to \mathbb{R}^t$ as follows. Let $\vec{x} = (x_1, ..., x_{t+1}) \in S^t$. Define $< z >= z(\vec{x}) = (z_0, z_1, ..., z_{t+1})$ by $z_0 = 0$ and $z_j = \sum_{i=1}^{j} x_i^2$ for $1 \le j \le t+1$. (Note $z_{t+1} = 1$). Let $m_j(i)$ be the measure of the $j^{th}$ color in $[z_{i-1}, z_i]$ and let $f_j(\vec{x}) = \sum_{i=1}^{t+1} sign(x_i) m_j(i)$. Let $f(\vec{x}) = (f_1(\vec{x}), ..., f_t(\vec{x})) \in \mathbb{R}^t$. Check. $f : S^t \to \mathbb{R}^t$ is continues and $f(\vec{x}) = -f(\vec{x})$. By Theorem 15.11, there exist $\vec{x}$ with $f(\vec{x}) = \vec{0}$.

Let $Z = \cup_i [z_{i-1}, z - i]$ over these $i$ with $sign(x_i) = +$. Since $f(\vec{x}) = \vec{0}$, we have $f_j(\vec{x}) = 0$ for all $j$, that is $Z$ captures exactly half of the measure of color $j$. This completes the proof of Lemma 15.8. ∎

To prove Lemma15.10, we first introduce two topological spaces. Let $k$ be an odd prime.

**Definition 15.12.** *Let $m \geq 1$. Let $X = X_{m,k}$ denote the CW-complex consisting of $k$ disjoint copies of the $m(k-1)$-dimensional balls with an identified boundary $S^{m(k-1)-1}$.*

Clearly, $S^n = \{(x_0, x_1, ..., x_n) \in \mathbb{R}^{n+1} : x_0^2 + ... + x_n^2 = 1\}$. Represent $S^{m(k-1)}$ as the set of all $m \times k$ real matrices $(a_{ij})$ satisfying $\sum_{j=1}^{k} a_{ij} = 0$ for all $q \leq i \leq m$ and $\sum_{i,j} a_{i,j}^2 = 1$.

**Definition 15.13.** *We define a fire action of the cyclic group $\mathbb{Z}_k$ on $X$, by define $w$, the action of its generator. Define $w(a_{ij}) \to (a_{i,j+1})$, where $j+1$ is module $k$. This is a free action, i.e. $w(x) \neq x$ for any $x \in S^{m(k-1)-1}$.*

Next, $w$ is extended from $S^{m(k-1)-1}$ to $X = X_{m,k}$ as follows. Let $(y, r, q)$ denote a point of $X$ from the $q^{th}$ ball with radius $r$ and $S^{m(k-1)-1}$-coordinate $y$. Then let $w(y, r, q) = (w(y), r, q+1)$, where $q+1$ is module $k$. Again $w$ is free action on $X$.

**Lemma 15.14** (Barany,shlosman and Lovasz)**.** *For any continues map $h : X_{m,k} \to \mathbb{R}^m$, there exists an $x \in X$ such that $h(x) = h(wx) = ... = h(w^{k-1}x)$.*

Let $N = (k-1)(m+1)$. Let $\Delta^N$ be the $N$-dimensional simplex, i.e. $\Delta^N = \{(x_0, x_1, ..., x_N) \in \mathbb{R}^{N+1} : \sum_{i=0}^{N} x_i = 1 \text{ and } x_i \geq 0\}$.

**Definition 15.15.** *The support of a point $x \in \Delta^N$ is the minimal face of $\Delta^N$ that contains $x$.*

Let $Y = Y_{N,k}$ denote the following CW-complex, $Y_{N,k} = \{(y_1, ..., y_k) : y_1, ..., y_k \in \Delta^N$ and the supports of the $y_i$'s are pairwise disjoint $\}$. Then there is an obvious free action $Z_k$ on $Y$: its generator $\gamma$ maps $(y_1, ....y_k)$ to $(y_2, ..., y_k, y_1)$.

**Definition 15.16.** *Let $T, R$ be two topological spaces. Let $\alpha, \beta$ be the free actions of $Z_k$ on $T, R$. We say a continues mapping $f : T \to R$ is $Z_k$-equivariant if $f \circ \alpha = \beta \circ f$.*

**Definition 15.17.** *The space $T$ is $s$-connected if for all $0 \leq \ell \leq s$, every continues map : $S^\ell \to T$ can be extended to a continues map : $B^{\ell+1} \to T$.*

**Lemma 15.18.** *Suppose $k$ is odd prime , $m \geq 1$, $N = (k-1)(m+1)$ and let $X = X_{m,k}, Y = Y_{N,k}$, $w$ and $\gamma$ be as before. Then $Y$ is $N-k = dim(X)-1$ connected, and thus there is a $Z_k$-qeuivariant map $f : X \to Y$.*

*p[roof of Lemma 15.10.* We can now prove Let $k$ be an odd prime and let $c$ be an interval $t$-coloring. Put $X = X_{(k-1)t,k}, Y = Y_{(k-1)t,k}$ and define a continuous function $g :: Y \to \mathbb{R}^{t-1}$ as follows. Let $y = (y_1, y_2, ..., y_k)$ be a point of $Y$. Recall that each $y_i$ is a point of $\Delta^N$, i.e., is an $(N+1)$-dimensional vector with nonnegative coordinates whose sum is 1, and that the supports of the $y_i$s are pairwise disjoint. Put $x = (x_0, ..., x_N) = 1/k(y_1 + y_2 + ... + y_k)$, and define a partition of $[0, l]$ into $N+1$ intervals $I_0, I_1, ..., I_N,$, where $I_0 = [0, x_0]$, $I_j = [\sum_{i=0}^{j-1} x_i, \sum_{i=0}^{j} x_i]$, $1 \leq j \leq N$. Notice that since the supports of the $y_\ell$'s are pairwise disjoint, if $x_j > 0$ (i.e., the interval $I_j$ has positive length), then there is a unique $\ell$, $1 \leq \ell \leq k$ such that the $j$-th coordinate of $y_l$, is positive. For $1 \leq \ell \leq k$, let $F_\ell$ be the family of all those $I_j$'s such that the $j$-th coordinate of y, is positive. Note that the sum of lengths of these $I_j$'s is precisely $l/k$. For $1 \leq i \leq t-1$, define $g_i(y)$ to be the measure of the $i$-th color in $\cup F_1$. Finally, put $g(y) = (g_1, g_2(y), ..., g_{t-1}(y))$. One can easily check that $g : Y \to \mathbb{R}^{t-1}$ is continuous. Moreover, for $1 \leq \ell \leq k$ and $1 \leq i \leq t-1$, $g - i(\gamma^{\ell-1}y)$ is the measure of the $i$-th color in $\cup F_\ell$. By Lemma 15.18, there exists a $Z_k$-equivariant map $f : x_{t-1,k} \to Y_{(t-1)k,k}$. Define $h = g_0 f : X \to Rt-1$. By

Lemma 15.14, there is some $x \in X$ such that $h(x) = h(wx) = ... = h(w^{k-1}x)$. By the equivariance of $f$, $y = f(x)$ satisfies $g(y) = g(\gamma y) = ... = g(\gamma^{k-1}y)$. But this means that each of the families of intervals $F_1, F_2, ..., F_k$ corresponding to $y$ captures precisely $l/k$ of the measure of each of the first $t-1$ colors. Since the total measure of each $F_j$ is $l/k$, each $F_j$ captures precisely $l/k$ of the measure of the last color, as well. Dividing the length 0 intervals arbitrarily between the $F_j$'s we conclude that there is a $k$-splitting of size $N = (k-1)t$, as desired. This completes the proof of Lemma 15.10. ∎

## References


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