Extremal and Probabilistic Graph Theory
Lecture 13
April 12st, Tuesday

Recall the lecture in last class:

- **Theorem 2.** If G is connected, then G has a path of size $\geq \min\{n, 2\delta(G)+1\}$.

  Tight: graph G consisting of $k_t^{,}s$ sharing a common vertex.

- **Theorem 3**(Erdős-Gallai).
$$ex(n, P_t) \leq \frac{(t-1)n}{2}.$$
  Moreover, when $\frac{t}{n}$, the extremal graph is unique, that is, a disjoint union of $k_t^{,}s$.

- **Posa' rotation.** Let P be a longest path in G from u to v. For $w \in N(v) \bigcap V(P)$, the path $P' = P - \{ww^+\} + \{wv\}$ is also a longest path in G. This transformation from P to P' is called a $Posa' rotation$.

New knowledge in this class:

- **Definition 1.** For $S \subseteq V(G)$, define $N(S) = \{v \notin S : v \in N(w) \text{ for } w \in S\}$ to be the neighborhood of S.

- **Theorem 6** (Posa). Suppose every $S \subseteq V(G)$ satisfying $|N(S)| \geq min\{n - |S|, 2|S| + 1\}$. Then G has a Hamiltonian path.

  Proof. Let P be a longest path in G from u to v. Let S be the set of all endpoint of paths obtained by repeatingly applying Posa's rotation from P, which preserve u as an endpoint. Define S^+, S^- (accroding to P). Then every vertex in S has no neighbors in $V(G) \backslash V(P)$, so $N(S) \subseteq V(P)$.

  Claim:$\forall\ x \in S$, $N_G(x) \subseteq S^+ \bigcup S \bigcup S^-$.

  Proof. Suppose $\exists\ y \in N(x)$, but $y \notin S^+ \bigcup S \bigcup S^-$. Then y^--y-y^+ is always a subgraph of any new path obtained by Posa's rotation. We then perform a rotation, using $xy \in E(G)$, to get a new path that ends at either y^+ or y^-, so y^+ or $y^- \in S$. Therefore $y \in S^-$ or $y \in S^+$, a contradiction. This proves claim. ∎

  Then $N(S) \subseteq S^+ \bigcup S^-$, so $|N(S)| \leq 2|S|$. But $|N(S)| \geq min\{n - |S|, 2|S| + 1\}$.

  This shows that $|N(S)| \geq n - |S|$, since $V(P) \geq N(S) \bigcup S$. Notice that $N(S)$ and S are disjoint, we have $|V(P)| \geq |N(S)| + |S| \geq n$, that is to say, P is Hamiltonian path. ∎

- **Remark.** This result is classical and has many application, that is to say, for random graph.

- **Definition 1.** A vertex u is a cut-vertex of G, if G-u is disconnected.

- **Menger' Theorem.** Let A, $B \subseteq V(G)$, G is connected if and only if $\exists$ 2 disjoint paths from A to B.

- **Theorem 7.** If G is 2-connected, then G has a cycle of length at least $\min\{n, 2\delta(G)\}$.

- **Remark 1.** Tight for graphs G consisting of some K_t's sharing 2 vertices.
- **Remark 2.** This implies Dirac's Theorem.
- **Exercise.** As the condition of Theorem 4 implies that such G must be 2-connected.
- **Proof of Theorem 7.** Let $P = x_0 x_1 \cdots x_m$ be a longest path in G. By Theorem 2, $|P| \geq min\{n-1, 2\delta(G)\}$.

  Case 0: $N(x_0) \bigcap N(x_m)^+ = \emptyset$.

  Proof of Case 0. Since $N(x_0)$, $N(x_m) \subseteq V(P)$. If $N(x_0) \bigcap N(x_m)^+ \neq \emptyset$, then we can form a cycle C, such that $V(C) = V(P)$. So $|C| = |P| + 1 \geq min\{n, 2\delta(G)+1\}$. Therefore, claim 1 is done. ∎

  Case 1: $\exists\ j < i-1$, such that $x_0 \sim x_i$, $x_m \sim x_j$.

  Proof of Case 1. Pick (i, j) such that $|i-j|$ is minimum. Let E' be the edge-set containing all edges in $x_j P x_i$. Let $C = (P - E') \bigcup \{x_0 x_i, x_j x_m\}$. We see that $N(x_0) \bigcup \{x_0\} \subseteq V(C)$ and $N(x_m)^+ \backslash \{x_j\}^+ \subseteq V(C)$. By claim 1, $N(x_0) \bigcap N(x_m)^+ = \emptyset$ and $x_0 \notin N(x_0) \bigcup N(x_m)^+$. Therefore $|C| \geq |N(x_0) \bigcup \{x_0\}| + |N(x_m)^+ \backslash \{x_j\}^+| \geq d(x_0) + 1 + d(x_m) - 1 \geq 2\delta(G)$. ∎

  Case 2: All neighbors of x_0 are before the neighbors of x_m. Let i be the maximum such that $x_i \sim x_0$. Let j be the minimum such that $x_j \sim x_m$. (Possibly $x_i = x_j$). Let $G_1 = G[V(x_0 P x_i)]$ and $G_2 = G[V(x_j P x_m)]$ which both are Hamiltonian.

  By Menger's Theorem, there are 2 disjoint paths Q_1, Q_2 from $V(G_1)$ to $V(G_2)$ in G, and we can choose Q_1, Q_2 such that x_i is an end of them, and x_j is an end of them.

  Why? If Q_1, Q_2 do not start at x_i, then we would begin traveling from x_i along $x_i P x_j$ until we encounter some P_i. Then we would define new paths Q_1, Q_2, one of which uses x_i as an end. Similarly, do this for x_j.

  Then we have the following two cases to consider: (a) x_i, $x_j \in Q_1$; (b) $x_i \in Q_1$, $x_j \in Q_2$. In both cases we can construct a cycle C using Q_1, Q_2 and all vertices in $\{x_0, x_m\} \bigcup N(x_0) \bigcup N(x_m)$, so $|C| \geq d(x_0) + d(x_m) + 2 - 1 \geq 2\delta(G) + 1$ (Note that the "-1" comes from the possibility that maybe $x_i = x_j$). This proves Theorem 7. ∎

- **Theorem 8**(Erdős-Gallai).

$$ex(n, \{C_{t+1}, C_{t+2}, \cdots\}) \leq \frac{t(n-1)}{2}.$$

  Moreover, it is tight when $n = k(t-1)+1$ for any k, where the extremal graphs are k copies of K_t's all sharing a fixed vertex.

  proof. By induction on n. This is trivial for $n \leq t$. Let $n > t$. Let G be a graph with no cycle of length $\geq t+1$.

  Case 1: $\delta(G) \geq \frac{t+1}{2}$.

  If G is 2-connected, then by theorem 7, $\exists$ a cycle $|C| \geq min\{n, 2\delta(G)\} \geq min\{n, t+1\} = t+1$. A contradiction. So G has a cut-vertex u such that $G_1 \bigcup G_2 = G$ and $V(G) \bigcap V(G_2) = \{u\}$. Let $n_i = |V(G_i)|$, then $n+1 = n_1 + n_2$.

  By induction, $e(G) = e(G_1) + e(G_2) \leq \frac{t(n_1-1)}{2} + \frac{t(n_2-1)}{2} = \frac{t(n-1)}{2}$. Case 1 is done.

  Case 2: $\exists\ v$ with $d(v) \leq \frac{t}{2}$. By induction. ∎