Extremal and Probabilistic Graph Theory
Lecture 14
April 19th, Tuesday

Recall ($Erd\ddot{o}s - Gallai$)**.**

$$ex(n, \{C_{t+1}, C_{t+2}, ...\}) \leq \frac{t(n-1)}{2}.$$

Definition 14.1 *A walk of length k in a graph G is an alternating sequence $(v_0, e_1, v_1, ..., v_{k-1}, e_k, v_k)$ where $v_i \in V(G)$ and $e_i = \{v_{i-1}, v_i\} \in E(G)$.*

Note. In this definition, a walk is an **ordered** sequence.

Definition 14.2 *A walk is non-backtracking if $e_i \neq e_{i+1}, 1 \leq i \leq k-1$.*

Definition 14.3 *Let $A = (a_{ij})_{n\times n}$ be the adjacency matrix of the graph G s.t*

$$a_{ij} = \begin{cases} 1 & \text{if } ij \in E(G), \\ 0 & \text{otherwise.} \end{cases}$$

Fact. For $u, v \in V(G)$ and the adjacency matrix A of G, $(A^k)_{uv}$ is the number of walks from u to v of length k in G.
Proof. Exercise (Hint: matrix multiplication and induction). ∎

Notation 14.4 *For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^n x_i y_i$.*

Theorem 14.5 *Let A be a symmetric matrix and let $\mathbf{x}$ be a unit vector of $\mathbb{R}^n$ s.t all entries of A and $\mathbf{x}$ are non-negative, then $\langle A^k\mathbf{x}, \mathbf{x} \rangle \geq \langle A\mathbf{x}, \mathbf{x} \rangle^k$.*

Proof. Since A is symmetric, there exists a matrix X s.t $D = X^{-1}AX = diag(\lambda_1, ..., \lambda_n)$ where λ_i is the eigenvalue of A and X consists of the eigenvectors $X_1, ..., X_n$ which form an orthonormal basis of $\mathbb{R}^n$. So $AX_i = \lambda_i X_i$. Let $\mathbf{x} = \sum_{i=1}^n a_i X_i$, where $\sum_{i=1}^n a_i^2 = 1$. Then $\langle A\mathbf{x}, \mathbf{x} \rangle = \sum_{i=1}^n \lambda_i a_i^2$. Since $A = XDX^{-1}$, we have $A^k = XD^kX^{-1}$ and $\langle A^k\mathbf{x}, \mathbf{x} \rangle = \sum_{i=1}^n \lambda_i^k a_i^2$. By Jensen's inequality($f(x) = x^k$ is convex),

$$\langle A^k\mathbf{x}, \mathbf{x} \rangle = \sum_{i=1}^n \lambda_i^k a_i^2 \geq (\sum_{i=1}^n \lambda_i a_i^2)^k = \langle A\mathbf{x}, \mathbf{x} \rangle^k.$$

Note. This proof only works when k is even as some λ_i^k may be negative when k is odd.
Case 1. $\mathbf{x}$ has an zero entry.

Without losing generality, say $x_1 = 0$. Let $\mathbf{y} \in \mathbb{R}^{n-1}$ be obtained from $\mathbf{x}$ by deleting x_1, and let $B \in \mathbb{R}^{(n-1)\times(n-1)}$ be obtained from A by deleting the 1st row and 1st column. By induction on n, $\langle B^k\mathbf{y}, \mathbf{y} \rangle \geq \langle B\mathbf{y}, \mathbf{y} \rangle^k$. Note that $\langle A\mathbf{x}, \mathbf{x} \rangle = \langle B\mathbf{y}, \mathbf{y} \rangle$, and $\langle A^k\mathbf{x}, \mathbf{x} \rangle \geq \langle B^k\mathbf{y}, \mathbf{y} \rangle$ (A exercise from linear algebra). So

$$\langle A^k\mathbf{x}, \mathbf{x} \rangle \geq \langle B^k\mathbf{y}, \mathbf{y} \rangle \geq \langle B\mathbf{y}, \mathbf{y} \rangle^k = \langle A\mathbf{x}, \mathbf{x} \rangle^k.$$

Case 2. All entries of $\mathbf{x}$ are positive.

By re-normalization, we may assume that the largest eigenvalue of A is $\lambda = 1$, with a non-negative unit eigenvector $\mathbf{z} \in \mathbb{R}^n$, i.e $A\mathbf{z} = \mathbf{z}$. (By $Perron - Frobenius$ Theorem).

If $\mathbf{x}$ is an eigenvector of A with $\lambda = 1$, then it is clear that

$$\langle A^k\mathbf{x}, \mathbf{x}\rangle = \langle \mathbf{x}, \mathbf{x}\rangle = 1 = \langle A\mathbf{x}, \mathbf{x}\rangle^k.$$

So we may assume that $\mathbf{x}$ is not such a thing. Then there exist a unit vector $\mathbf{y} \in \mathbb{R}^n$ which is orthogonal to $\mathbf{z}$ s.t $\mathbf{x} = \alpha\mathbf{z} + \beta\mathbf{y}$, with $\alpha^2 + \beta^2 = 1$, where $\alpha, \beta \geq 0(\mathbf{x}, \mathbf{z}$ are non-negative).

Claim 1. $\langle A\mathbf{y}, \mathbf{y}\rangle < 1$.
Proof. Suppose that $\langle A\mathbf{y}, \mathbf{y}\rangle = 1$. Then $\langle A\mathbf{x}, \mathbf{x}\rangle = \alpha^2\langle A\mathbf{z}, \mathbf{z}\rangle + \beta^2\langle A\mathbf{y}, \mathbf{y}\rangle = \alpha^2 + \beta^2 = 1$. But $\langle A\mathbf{x}, \mathbf{x}\rangle = \sum_{i=1}^n \lambda_i a_i^2 \leq \max\{\lambda_1, ..., \lambda_n\} = 1$. This implies $\lambda = 1$, and $\mathbf{x}$ is an eigenvector of $\lambda = 1$. This is a contradiction. ∎

We can select a non-negative unit vector $\mathbf{w} \in \mathbb{R}^n$ s.t $\mathbf{w}$ has a zero entry and $\mathbf{w} = \alpha'\mathbf{z} + \beta'\mathbf{y}$, where $0 \leq \alpha' \leq \alpha, 0 \leq \beta', \alpha'^2 + \beta'^2 = 1$. By Case 1, we have $\langle A^k\mathbf{w}, \mathbf{w}\rangle \geq \langle A\mathbf{w}, \mathbf{w}\rangle^k$.

Let $f : [0, 1] \to \mathbb{R}$ by

$$f(t) = 1 - t^k + \frac{\langle A^k\mathbf{y}, \mathbf{y}\rangle - 1}{\langle A\mathbf{y}, \mathbf{y}\rangle - 1}(t - 1).$$

Then $f'' \leq 0$, which means f is convex on $[0, 1]$. And $f(1) = 0$.

Claim 2. $f(\langle A\mathbf{x}, \mathbf{x}\rangle) = \langle A^k\mathbf{x}, \mathbf{x}\rangle - \langle A\mathbf{x}, \mathbf{x}\rangle^k$. Similarly, $f(\langle A\mathbf{w}, \mathbf{w}\rangle) = \langle A^k\mathbf{w}, \mathbf{w}\rangle - \langle A\mathbf{w}, \mathbf{w}\rangle^k$.
Proof. It is the same to verify

$$\frac{\langle A^k\mathbf{y}, \mathbf{y}\rangle - 1}{\langle A\mathbf{y}, \mathbf{y}\rangle - 1} = \frac{\langle A^k\mathbf{x}, \mathbf{x}\rangle - 1}{\langle A\mathbf{x}, \mathbf{x}\rangle - 1}.$$

Hint:$\langle A\mathbf{x}, \mathbf{x}\rangle = \alpha^2\langle A\mathbf{z}, \mathbf{z}\rangle + \beta^2\langle A\mathbf{y}, \mathbf{y}\rangle = 1 + \beta^2(\langle A\mathbf{y}, \mathbf{y}\rangle - 1)$. Similarly, $\langle A^k\mathbf{x}, \mathbf{x}\rangle = 1 + \beta^2(\langle A^k\mathbf{y}, \mathbf{y}\rangle - 1)$. ∎

Claim 3. $\langle A\mathbf{w}, \mathbf{w}\rangle \leq \langle A\mathbf{x}, \mathbf{x}\rangle \leq 1$.
Proof. We have $\langle A\mathbf{x}, \mathbf{x}\rangle = 1 + \beta^2(\langle A\mathbf{y}, \mathbf{y}\rangle - 1)$. $\langle A\mathbf{w}, \mathbf{w}\rangle = 1 + \beta^2(\langle A\mathbf{y}, \mathbf{y}\rangle - 1)$. Since $\beta^2 \leq \beta'^2$, by Claim 1, we can prove $\langle A\mathbf{w}, \mathbf{w}\rangle \leq \langle A\mathbf{x}, \mathbf{x}\rangle \leq 1$. ∎

By Claim 2 and Claim 3 we have

$$\langle A^k\mathbf{x}, \mathbf{x}\rangle - \langle A\mathbf{x}, \mathbf{x}\rangle^k = f(\langle A\mathbf{x}, \mathbf{x}\rangle) \geq \min\{f(\langle A\mathbf{w}, \mathbf{w}\rangle), f(1) = 0\} \geq 0.$$

Now the proof of theorem 14.5 is finished. ∎

Corollary 14.6 *A graph G of average degree d has at least nd^k walks of length k.*

Proof. Take A as the adjacency matrix of G, and $\mathbf{x} = \frac{1}{\sqrt{n}}\mathbf{1}$. By theorem 1, $\langle A^k\mathbf{x}, \mathbf{x}\rangle \geq \langle A\mathbf{x}, \mathbf{x}\rangle^k$. Here, $\langle A\mathbf{x}, \mathbf{x}\rangle = 2e/n = d$, and $\langle A^k\mathbf{x}, \mathbf{x}\rangle = \#\{k - walks\}/n$, this implies $\#\{k - walks\} \geq nd^k$. This is tight for any d-regular graphs(easy to verify). ∎

Theorem 14.7 *Let G be a n-vertex graph with average degree d, where the smallest degree of vertices is at least 2($\delta(G) \geq 2$). Then G has at least $nd(d-1)^{k-1}$ non-backtracking walks of length k with equality iff G is a d-regular graph.*

Proof. In next lecture. ∎

Here we introduce some definitions related to theorem 14.7.

Definition 14.8 *Consider each edge $uv \in E(G)$ as a pair of directed edges $\overrightarrow{uv}$ and $\overrightarrow{vu}$. Let A be a $nd \times nd$ matrix indexed by all directed edges.*

$$(A)_{\overrightarrow{uv},\overrightarrow{wz}} = \begin{cases} 1 & \textit{if } u \neq z \textit{ and } v = w, \\ 0 & \textit{otherwise.} \end{cases}$$

Definition 14.9 *Let P be defined by*

$$(P)_{\overrightarrow{uv},\overrightarrow{wz}} = \begin{cases} 1/(d(v)-1) & \textit{if } u \neq z \textit{ and } v = w, \\ 0 & \textit{otherwise.} \end{cases}$$

The matrix P is the transition matrix of a random walk. Let $\mathbf{x} = \frac{1}{nd}\mathbf{1} \in \mathbb{R}^{nd}$, then $\mathbf{x}P = \mathbf{x}$.

Definition 14.10 *A $non-backtracking$ random walk is defined as following: If at step t, the walk is at the edge $\overrightarrow{uv}$, then at step $t+1$, we move to an edge $\overrightarrow{vz}$, where z is a random neighbor of v(but not u).*