Extremal and Probabilistic Graph Theory
April 5th, Tuesday

- **Definition 1.** A Kneser graph $K_{n,t}$ is a graph with vertex set $v = \binom{[n]}{t}$ and edge set $\{(A, B) : A, B \in \binom{[n]}{t}, A \bigcap B = \phi\}$, where $n \geq 2t + 1$.
- **Theorem(Lovasz)**

$$\chi(K_{n,t}) = n - 2t + 2$$

Corollary: when $n = 3t - 1$, $\chi(K_{n,t}) = t + 1$ and $K_{n,t}$ is triangle-free.

This show that there exists triangle-free graph $G$ with arbitrary large chromatic number.

- **Example:** $K_{5,2}$ Peterson graph $P$ with $\chi(p) = 3$, which is not edge-critical.

Mantel's Theorem shows that an $n$-vertex triangle-free graph $G$ with $\delta(G) \geq \frac{n}{2}$ must be bipartite.

On the other hand, not all triangle-free graphs are bipartite.($C_5$)

- **Andrásfai-Erdős-Sós Theorem.** All triangle-free graph of mini-degree $> \frac{2n}{5}$ are bipartite. Moreover there is a non-bipartite triangle-free graph with mini-degree=$\lfloor \frac{2n}{5} \rfloor$.

**Proof.** The blow-up $C_5(\frac{n}{5})$ is the example:

$$\lfloor \frac{n}{5} \rfloor \leq |V_1| \leq ... \leq |V_5| \leq \lceil \frac{n}{5} \rceil$$

Given a triangle-free $G$ with $\delta(G) > \frac{2n}{5}$, suppose for a contradiction that $G$ is not bipartitie.

Then G contains an odd cycle.

Let $C$ be a shortest odd cycle in $G$.$\Rightarrow |C| \geq 5$

**Case 1:** $|C| \geq 7$

For $v_i, v_j \in V(C)$ of distance at least 3 on $C$, then $N(v_i) \bigcap N(v_j) = \phi$ (otherwise, it gives a shorter odd cycle).

For adjacent $v_i, v_j \in V(C)$, we also have $N(v_i) \bigcap N(v_j) = \phi$ (otherwise, there exists triangle).

Pick any $v_1 \in V(C)$ and $v_i, v_j \in V(C)$ which are of distance 3 to $v_1$.

So any 2 of $v_1, v_i, v_j$ have no common neighbours.

$\Rightarrow N(v_1), N(v_i), N(v_j)$ are disjoint

$$n = |V(G)| \geq |N(v_1)| + |N(v_i)| + |N(v_j)| > \frac{6n}{5}$$

a contradiction.

**Case 2:** $|C| = 5$

Let $C = v_1 - v_2 - v_3 - v_4 - v_5 - v_1$

For any $i$, $N(v_i) \bigcap N(v_{i+1}) = \phi$.

$|N(v_i)\bigcup N(v_{i+1})| > \frac{4n}{5}$.

Consider $|w_i| = |N(v_{i-1})\bigcap N(v_{i+1})| = |N(v_{i-1})| + |N(v_{i+1})| - |N(v_{i-1})\bigcup N(v_{i+1})|$

$\geq |N(v_{i-1})| + |N(v_{i+1})| - |n - N(v_i)|$

$> 3 \times \frac{2n}{5} - n = \frac{n}{5}$.

Also we notice that $w_1, ..., w_5$ are pairwise disjoint.

Then $|\bigcup_{i=1}^{5} w_i| > n$, a contradiction. ∎

- **Definition:** The chromatic threshold of a graph $H =$ infimum $\delta$ s.t. every n-vertex $H$-free graph $G$ of min-degree at least $\delta n$ has

$$\chi(G) \leq F(\delta, H)$$

  i.e. $\chi(G)$ is bounded.

- **Theorem 1. (Thomassen)**

$$ct(K_3) = \frac{1}{3}$$

- **Theorem 2. (Goddord and Lyle)**

$$ct(K_r) = \frac{2r-5}{2r-3}$$

- **Definition:** A graph $H$ with $\chi(H) = r$ is r-near-acyclic if we can delete r-3 color classes from $H$ to obtain a graph $H'$ s.t. $V(H') = X\bigcup Y$ where $X$ is independent and all edges of $H'$ in $Y$ are vertex-disjoint.(matching)

- **Theorem:** Let H be a graph with $\chi(H) = r \geq 3$.

  Then $ct(H) \in \{\frac{r-3}{r-2}, \frac{2r-5}{2r-3}, \frac{r-2}{r-1}\}$,where

. $ct(H) = \frac{r-3}{r-2}$ iff H is r-near-acyclic.

. $ct(H) = \frac{2r-5}{2r-3}$ iff H is not r-near-acyclic and $\mu(H)$ contains a forest.

. $ct(H) = \frac{r-2}{r-1}$ iff $\mu(H)$ has no forest.

- **Theorem. (Brandt and Thomassé)** Triangle-free $G$ with $\delta(G) > \frac{n}{3}$ has $\chi(G) \leq 4$.

Peterson graph is 3-near-acyclic $\Rightarrow ct(P) = 0$.