Extremal and Probabilistic Graph Theory
March 10

- **Lemma(Erdős-Moon).** Let$G$ be an $n$-vertex graph of edge density $p$ $s.t$

$$e(G) = p\binom{n}{2} \geq \frac{1}{2}s^{1+\frac{1}{s}}n^{2-\frac{1}{s}} + 2sn.$$

Then, #{copies of $K_{s,s}$ in $G$}$\geq \Omega(p^{s^2}n^{2s})$.

- **Remark.** This is a quantified version of supersaturation lemma for $K_{s,s}$.

- **Proof.** Let $M$ =#{pair $(v, S)$ where $S \subset N(v)$}. Obviously,

$$M = \sum_{v \in V(G)} \binom{d(v)}{s}.$$

$\forall$ subsets $S$ of size $s$, let $f(s)$ be the number of vertices $v$ $s.t$ $S \subset N(v)$. We have

$$M = \sum_{S \in \binom{V}{s}} f(s).$$

Noting that

$$\frac{M}{n} = \frac{\sum_{v \in V(G)} \binom{d(v)}{s}}{n} \geq \binom{\frac{\sum d(v)}{n}}{s} = \Omega((pn)^s),$$

we have $\sum_{S \in \binom{V}{s}} f(s) = \Omega(p^s n^{s+1})$. On the other hand, #{copies of $K_{s,s}$ in $G$}= $\frac{1}{2}\sum_{S \in \binom{V}{s}} \binom{f(s)}{s}$,

$$\frac{1}{2}\sum_{S \in \binom{V}{s}} \binom{f(s)}{s} \geq \frac{1}{2}\binom{n}{s}\binom{\frac{\sum_{S \in \binom{V}{s}} f(s)}{\binom{n}{s}}}{s} \geq \Omega(1)n^s(p^s n)^s = \Omega(p^{s^2}n^{2s}).$$

- **Question.** Why do we use the condition of this lemma?

- **Theorem 1.** For $t \geq 2$ and $k$, $\exists$ constant $C_k(t) > 0$ $s.t$ the following holds:
Any k-graph with $e(G) = \binom{n}{k} \geq C_k(t)n^{k-(\frac{1}{t})^{k-1}}$ has at least $\Omega(p^{t^k}n^{tk})$ copies of $K_{t:k}$.

- **Remark.** Case $k = 2$ is just the Erdős-Moon.

- **Theorem 2(K-S-T for hypergraph).** For $t \geq 2$,

$$ex_k(n, K_{t:k}) = O(n^{k-(\frac{1}{t})^{k-1}}).$$

- **Remark 1.** This implies $\pi(K_{t:k}) = 0$.

- **Remark 2.** Theorem 1 can imply theorem 2.

- **Proof of Theorem 2.** Assuming Theorem 1 holds for $k-1$, we suppose there is a $K_{t:k}$-free k-graph $G$ with $e(G) = \omega(n^{k-(\frac{1}{t})^{k-1}})$.

  (**Note** :$m(n) = \omega(n)$ means $m(n)/n$ has a sufficiently large lower bound for sufficiently large $n$)

  **Recall** : Let $H$ be a k-graph with $(d-1)n+t$ edges, then $H$ has a subgraph $J$ with $\delta(J) \geq d$ and $|V(J)| \geq t^{\frac{1}{k}}$.
  $\Rightarrow \exists$ a subgraph of $G$ with $\delta(J) \geq \omega(n^{k-1-(\frac{1}{t})^{k-1}})$ which is much larger than $O((n^{k-1-(\frac{1}{t})^{k-2}}))$.
  Then the link hypergraph $J_v$ for $v \in V(J)$ is a (k-1)-graph with

  $$\delta(J) \geq \omega(n^{k-1-(\frac{1}{t})^{k-1}}) \geq \omega(m^{k-1-(\frac{1}{t})^{k-1}})$$

  where $m = |V(J_v)|$. By Theorem1, $J_v$ has

  $$\Omega((\frac{d(v)}{\binom{m}{k-1}})t^{k-1}m^{t(k-1)}) = \omega(m^{t(k-1)-1})$$

  copies of $K_{t:(k-1)}$.
  $\Rightarrow \exists\ \omega(m^{t(k-1)})$ copies of $(v, K)$ where $K = K_{t:(k-1)} \subset J_v$.
  By Pigeonhole Principle, $\exists$ a fixed $K = K_{t:(k-1)}$ and $v_1...v_t \in V(J)\ s.t\ K \subset J_{v_i}\ \forall\ i$
  $\Rightarrow \exists\ K_{t:k} = K \cup \{v_1...v_t\}$ which is a contradiction. This proves Theorem 2. ∎

- **Proof of Theorem 1.** By induction on $k$. Base case $k = 2$ is just the Erdős-Moon Theorem.

  Suppose it holds for (k-1)-graphs. Given a k-graph $G$ with $e(G) = p\binom{n}{k} = \Omega(n^{k-(\frac{1}{t}^{k-1})})$ .
  Let $V_1 = \{v \in V : d(v) \geq C_k(t)n^{k-1-(\frac{1}{t})^{k-2}}\}$ and $V_2 = V \backslash V_1$. Since

  $$\sum_{v \in V_2} d(v) \leq O(n^{k-(\frac{1}{t})^{k-2}}) << n^{k-(\frac{1}{t})^{k-1}} \approx e(G),$$

  almost all edges of $G$ are in $V_1$.
  For $S \in \binom{V(G)}{t...t}$ where $t...t$ contains $(k-1)$ $t's$, Let
  $f(S) = \#\{v \in V(G) : v \cup S$ induces a $K^{(k)}_{1,t...t}$ in $G\}$ where $t...t$ contains $(k-1)$ $t's$.

  For $v \in V_1$, the link hypergraph $G_v$ is a (k-1)-graph with $d(v) \geq C_k(t)n^{k-1-(\frac{1}{t})^{k-2}}$ edges. By induction on $k-1$ for $G_v$, $G_v$ has

  $$\Omega((\frac{d(v)}{\binom{n}{k-1}})t^{k-1}n^{t(k-1)}) = \Omega((d(v))^{t^{k-1}}n^{(k-1)t-(k-1)t^{k-1}})$$

  (k-1)-graphs.

  **Claim** : $\#\{K^{(k)}_{1,t...t}$ in $G\}$

  $$\begin{aligned} &\geq \sum_{v \in V_1(G)} \Omega((d(v))^{t^{k-1}}n^{(k-1)t-(k-1)t^{k-1}}) \\ &\geq n(\frac{\sum d(v)}{n})^{t^{k-1}}\Omega(n^{(k-1)t-(k-1)t^{k-1}}) \\ &= e(G)^{t^{k-1}}\Omega(n^{(k-1)t-(k-1)t^{k-1}+1-t^{k-1}}) \\ &= p^{t^k}\Omega(n^{(k-1)t+1}) \\ &= \Omega(p^{t^k}n^{(k-1)t+1}) \end{aligned}$$

where $t...t$ contains $(k-1)$ $t's$.

On the other hand,$\#\{K^{(k)}_{1,t...t}$ in $G\}$=$\sum_{S\in\binom{V(G)}{t...t}} f(S)$. Therefore $\#\{K_{t:k}$ in $G\}$=

$$\begin{aligned}\frac{1}{k}\sum_{S\in\binom{V}{t...t}}\binom{f(s)}{t} &\geq \Omega(n^{t(k-1)})\frac{\sum f(S)^t}{|\binom{V}{t...t}|} \\ &\geq \Omega(n^{t(k-1)})(\frac{\sum f(S)}{n^{t(k-1)}})^t \\ &\geq \Omega(n^{t(k-1)})(p^{t^{k-1}}n)^t \\ &= \Omega(p^{t^k}n^{tk})\end{aligned}$$

where $t...t$ contains $(k-1)$ $t's$.

- **Question.** Why do we need $e(G)\geq\Omega(n^{k-(\frac{1}{t})^{k-1}})$?
- **Remark.** Theorem1 does imply Theorem2. Let $p\approx n^{-(\frac{1}{t})^{k-1}}$ so that $p\binom{n}{k}\geq(n^{k-(\frac{1}{t})^{k-1}})$, then Theorem1 gives $\Omega(p^{t^k}n^{tk})=\Omega(n^{tk-t})$ copies of $K_{t:k}$. This proves Theorem 1. ∎