Extremal and Probabilistic Graph Theory
March 3

- **Definition.** F is called degenerate k-graph if $\pi(F) = 0$.
- **Definition.** A k-graph G is $\mathcal{F}$-free, if G has NO $F \in \mathcal{F}$ as a subgraph, where $\mathcal{F}$ is a family of k-graphs.
- **Problem.** Characterize $\mathcal{F}$ with $\pi(\mathcal{F}) = 0$.
- **Kövari-Sós-Turán Theorem**$(k = 2)$**.** For $\forall t \geq s \geq 2$,

$$ex(n, K_{s,t}) \leq \frac{1}{2}(t-1)^{\frac{1}{s}} n^{2-\frac{1}{s}} + \frac{1}{2}(s-1)n.$$

Proof. Let G be a $K_{s,t}$-free n-vertex graph, we count T =the number of s-stars in G. On the one hand,

$$T = \sum_{v \in V(G)} \binom{d_v}{s},$$

On the other hand,

$$T \leq \sum_{S \in \binom{V}{S}} (t-1) = (t-1)\binom{n}{s}.$$

Define

$$f(x) = \begin{cases} 0 & x < s; \\ \binom{x}{s} & x \geq s, \end{cases}$$

Then by The Jensen Inequality,

$$\frac{\sum_{v \in V(G)} \binom{d_v}{s}}{n} \geq \binom{\frac{\sum d_v}{n}}{s} = \binom{\frac{2e(G)}{n}}{s} \geq \frac{(d-s+1)^s}{s!},$$

where $d = \frac{2e(G)}{n}$. So

$$\frac{(d-s+1)^s}{s!} \leq \frac{1}{n}(t-1)\binom{n}{s} \leq \frac{1}{n}(t-1)\frac{n^s}{s!},$$

and

$$d \leq (t-1)^{\frac{1}{s}} n^{1-\frac{1}{s}} + (s-1),$$

thus

$$e(G) \leq \frac{1}{2}(t-1)^{\frac{1}{s}} n^{2-\frac{1}{s}} + \frac{1}{2}(s-1)n.$$

∎

- **Remark.** For any bipartite G, there $\exists s$ and t such that $G \subseteq K_{s,t}$, but a G-free graph must be a $K_{s,t}$-free graph, so $ex(n, G) \leq ex(n, K_{s,t})) \leq O(n^{2-\frac{1}{s}})$.

- **Zarankiewicz Problem.** Let $Z(m, n, s, t)$ be the maximum value of $e(G)$, where G is a bipartite graph with two parts of size m and n, and G is $K_{s,t}$-free, then compare $ex(n, K_{s,t})$ and $Z(\frac{n}{2}, \frac{n}{2}, s, t)$.
- **Exercise.** $Z(\frac{n}{2}, \frac{n}{2}, s, t) \le ex(n, K_{s,t}) \le 2Z(\frac{n}{2}, \frac{n}{2}, s, t)$.
- **Exercise.** Find an upper-bound of $Z(n, n, s, t)$.
- **Theorem 1.** A family $\mathcal{F}$ of graphs has $\pi(\mathcal{F}) = 0$ iff $\mathcal{F}$ contains a bipartite graph.
  Proof. ($\Leftarrow$) Let $F \in \mathcal{F}$ be a bipartite graph, then there exists s such that $F \subseteq K_{s,s}$, a F-free graph necessarily is $K_{s,s}$-free, so

  $$ex(n, \mathcal{F}) \le ex(n, K_{s,s}) \le O(n^{2-\frac{1}{s}}),$$

  then

  $$\pi(\mathcal{F}) = 0.$$

  ($\Rightarrow$) Consider $\mathcal{F}$ with $\pi(\mathcal{F}) = 0$. Suppose $\mathcal{F}$ has NO bipartite graph, then $K_{\frac{n}{2},\frac{n}{2}}$ must be $\mathcal{F}$-free, so

  $$ex(n, \mathcal{F}) \ge e(K_{\frac{n}{2},\frac{n}{2}}) = \frac{n^2}{4},$$
  $$\pi(\mathcal{F}) \ge \frac{1}{4},$$

  which is a contradiction. ∎
- **Theorem 2.** For $\forall t, k$, $\pi(K_{t:k}) = 0$.
  Proof. We prove it by induction on k.
  When $k = 2$, $\pi(K_{t,t}) = 0$ by K-S-T theorem.
  Recall the supersaturation lemma: Let F be a k-graph, $\forall \varepsilon$, $\exists \delta > 0$, if G has at least $ex_k(n, F) + \varepsilon n^k$ edges, then G has at least $\delta n^{|V(F)|}$ copies of F.
  Claim: Let H be a k-graph with $(d-1)n + t$ edges, then H has a subgraph J with min-degree $\delta(J) \ge d$ and $|V(J)| \ge t^{\frac{1}{k}}$.
  Proof of claim: We prove it by greedy algorithm. Let $H_0 = H$, suppose now we have subgraph H_i, if H_i has a vertex v_i with degree$\le d-1$, then delete v_i, and let $H_{i+1} = H_i - v$, otherwise $\delta(H_i) \ge d$ and we stop. Let H_m be the subgraph it stops at, let $J = H_m$, then

  $$e(J) = e(H) - \sum_{j=0}^{m-1} d_{H_i}(v_i) \ge e(H) - m(d-1) \ge t.$$

  Note that we already have $\delta(J) \ge d$, now $|V(J)|^k \ge e(J) \ge t$, so $|V(J)| \ge t^{\frac{1}{k}}$. ∎
  Suppose that $\pi(K_{t:k-1}) = 0$, now we want to show $\pi(K_{t:k}) = 0$.
  For $\forall \varepsilon > 0$ and n large enough, let G be a $K_{t:k}$-free n-vertex k-graph, we want to show $e(G) \le \varepsilon n^k$. Suppose for a contradiction that $e(G) \ge \varepsilon n^k$, by claim, G has a subgraph J such that

  $$\delta(J) \ge \frac{\varepsilon}{2} n^{k-1}$$
  $$m \triangleq |V(J)| \ge (\frac{\varepsilon}{2})^{\frac{1}{k}} n$$

For $\forall v \in V(J)$, consider the link hypergraph J_v of v, then J_v is a $(k-1)$-graph with $(m-1)$ vertices and at least $\frac{\varepsilon}{2}n^{k-1}$ edges. By $\pi(K_{t:k-1}) = 0$ and the supersaturation lemma, we know that

$$\begin{aligned} e(J_v) &\geq \frac{\varepsilon}{2}n^{k-1} \geq \frac{\varepsilon}{2}m^{k-1} \\ &\geq ex_{k-1}(m, K_{t:k-1}) + \frac{\varepsilon}{4}m^{k-1}. \end{aligned}$$

So J_v has at least $\delta m^{(k-1)t}$ copies of $K_{t:k-1}$, $\forall v \in V(J)$, then
$\#\{(v, K) : K$ is a copy of $K_{t:(k-1)}$ in $J_v\} \geq \delta m^{1+(k-1)t}$.
For a fix subset X of size $(k-1)t$, we have N many ways to partition X into $k-1$ parts of size t, where

$$N = \binom{(k-1)t}{t, \ldots, t} = \frac{[(k-1)t]!}{t! \ldots t!}.$$

By pigeonhole principle, there $\exists$ a fixed $K = K_{t,(k-1)}$ such that there are at least $\frac{\delta m}{N}$ vertices belonging to $\{(v, K)\}$. Since $\frac{\delta m}{N} \gg t$, we can find $v_1, \ldots, v_t$ such that $K \subseteq J_{v_i}$ for $\forall i$, thus $G[\{v_1, \ldots, v_t\} \cup V(K)]$ is a $K_{t:k} \subseteq G$, but G is $K_{t:k}$-free, this is a contradiction. So for large n,

$$e(G) \leq \varepsilon n^k,$$

$$e_k(n, K_{t:k}) \leq \varepsilon n^k,$$

then $\forall \varepsilon \geq 0$,

$$\pi(K_{t:k}) = \lim_{n \to \infty} \frac{e_k(n, K_{t:k})}{n^k} \leq \varepsilon,$$

so $\pi(K_{t:k}) = 0$. ∎