Lecture Notes

RECALL

Theorem 1. Let $\mathcal{F}$ be a family of graphs, then $\pi(\mathcal{F}) = 0$ if and only if $\mathcal{F}$ contains a bipartite graph.

Theorem 2. (Assuming Supersaturation Lemma.) $\pi(K_{t:k}) = 0$ for any k, t.

We shall prove a stronger result than Theorem 2 later.

Supersaturation Lemma. Fix $k \geqslant 2$ and let F be a k-graph. For any $\epsilon > 0$, there exists $\delta > 0$, such that for any k-graph H on n vertices, if H has at least $\mathrm{ex}_k(n, F) + \epsilon n^k$ edges, then H contains at least $\delta\binom{n}{v(F)}$ copies of F, where $v(F)$ is the number of vertices of F.

Proof of Supersaturation Lemma. By definition of $\pi(F)$, we can find and fix an integer m, such that $\mathrm{ex}_k(m, F) \leqslant (\pi(F) + \frac{\epsilon}{2})\binom{m}{k}$. Let $\mathcal{C} = \left\{M \in \binom{V(G)}{m} : e(G[M]) > (\pi(F) + \frac{\epsilon}{2})\binom{m}{k}\right\}$. Then, we have

$$\begin{aligned} \#\{(e, M) : e \in G[M]\} &= \sum_{e \in E(G)} \binom{n-k}{m-k} \\ &\geqslant (\pi(F) + \epsilon)\binom{n}{k}\binom{n-k}{m-k} \\ &= (\pi(F) + \epsilon)\binom{n}{m}\binom{m}{k}. \end{aligned}$$

On the other hand,

$$\begin{aligned} \#\{(e, M) : e \in G[M]\} &= \sum_{M \in \binom{V(G)}{m}} e(G[M]) \\ &= \sum_{M \in \mathcal{C}} e(G[M]) + \sum_{M \notin \mathcal{C}} e(G[M]) \\ &\leqslant |\mathcal{C}|\binom{m}{k} + (\binom{n}{m} - |\mathcal{C}|)(\pi(F) + \frac{\epsilon}{2})\binom{m}{k}. \end{aligned}$$

Combining these two inequalities, we can get

$$\frac{\epsilon}{2}\binom{n}{m} \leqslant (1-\pi(F)-\frac{\epsilon}{2})|\mathcal{C}| \leqslant |\mathcal{C}|.$$

That is, G has at least $\frac{\epsilon}{2}\binom{n}{m}$ m-sets M with $e(G[M]) > (\pi(F)+\frac{\epsilon}{2})\binom{m}{k} \geqslant \mathrm{ex}_k(m,F)$. So each such M contains a copy of F. Since we have at least $\frac{\epsilon}{2}\binom{n}{m}$ such M's and each copy is contained at most $\binom{n-v(F)}{m-v(F)}$ such M's, by Pigeonhole Principle, the number of F-copies in G is at least

$$\frac{\epsilon}{2}\binom{n}{m}/\binom{n-v(F)}{m-v(F)} = \delta\binom{n}{v(F)}.$$

Here $\delta \triangleq \epsilon\binom{m}{v(F)}/2$ is a constant independent of n. □

Definition 1. *A k-graph H is k-partite if $V(H) = V_1\dot\cup\cdots\dot\cup V_k$ such that $|e\cap V_i| = 1$ for any $e \in E(H)$ and all $i = 1,\cdots,k$.*

Definition 2. *A cut of a k-graph H is a k-partite subgraph which contains all vertices of H.*

Fact 1. *Any k-graph H contains a cut with at least $\frac{k!}{k^k}e(H)$ edges.*

Proof. Exercise. □

Definition 3. *A cut with partition $V_1\dot\cup\cdots\dot\cup V_k$ is balanced if $\big||V_i|-|V_j|\big| \leqslant 1$ for all $i,j = 1,\cdots,k$.*

Fact 2. *Any k-graph H has a balanced cut with at least $\frac{k!}{k^k}e(H)$ edges.*

Proof. Exercise. □

Theorem 3. Let $\mathcal{F}$ be a family of k-graphs, then $\pi(\mathcal{F}) = 0$ if and only if $\mathcal{F}$ contains a k-partite k-graph.

Proof. Exercise. (Hint: it's generlization of Theorem 1) □

Definition 4. *For a k-graph F, the t-blowup $F(t)$ of F is a k-graph obtained from F by replacing each vertex $v \in V(F)$ with t copies, say $x_v^1,\cdots,x_v^t$ and by adding all edges $x_{v_1}^{a_1},\cdots,x_{v_k}^{a_k}$ into $F(t)$ for any edge $v_1\cdots v_k \in E(F)$ and for all $1 \leqslant a_1,\cdots,a_k \leqslant t$.*

Blowup Theorem. For any k-graph F and for all integer $t \geqslant 1$, $\pi(F(t)) = \pi(F)$.

Proof. Note that $F \subset F(t)$, so $\mathrm{ex}_k(n, F) \leqslant \mathrm{ex}_k(n, F(t))$ for all integer n, and as a consquence, we have $\pi(F) \leqslant \pi(F(t))$.

It remains to show $\pi(F(t)) \leqslant \pi(F)$.

Suppose it is not ture, then by definition of Turán density, there exist $\epsilon > 0$ and an $F(t)$-free k-graph G on surfficiently large n vertices, with $e(G) > (\pi(F) + \epsilon)\binom{n}{k}$ edges. We will find a copy of $F(t)$ in G to get a contradiction.

By supersaturation lemma, there exists some constant $\delta > 0$ and G contains at least $\delta\binom{n}{v(F)}$ copies of F.

Next we define an anxiliary $v(F)$-graph H on $V(G)$ as follows. For any $X \in \binom{n}{v(F)}$, X is an edge of H if and only if $G[X]$ contais a copy of F. So H has at least $\delta\binom{n}{v(F)}/v(F)!$ edges. As a consquence, H contains a copy of $K \triangleq K_{T;v(F)}$ since $\pi(K_{T;v(F)}) = 0$. Here T is choosen to be a large constant independent of n.

Let us fix a linear ordering of F, say $x_1, \cdots, x_{v(F)}$. Note that each copy of F, say F', induces one of the $v(F)!$ mappings like $\pi_{F'} : V(F') \to V(F)$. So we can color the edges of K by $v(F)!$ colors, depending on the mappings induced by edges. Now Ramsey Theorem implies that for large T, K has a monochromatic copy of $K_{t;v(F)}$, say K'. Then $G[V(K')]$ contains a copy of $F(t)$. It's a contradiction. $\square$

Exercise. Why this type of Ramsey number is finite?