Extremal And Probabilistic Graph Theory
May 5th Thursday

Theorem 1. Let G be a graph with average degree $d(G)$ and $g(G) \geq 2q+1$, then G has cycles of $\Omega(d^g)$ consecutive even length.

(We use three lemmas.)

Lemma 1. Let G be $s.t.$ $\delta(G) \geq 6(d+1)$, and girth $\geq 2q+1$, then for $\forall X \subseteq V(G)$ with $|X| \leq \frac{1}{3}d^g$, we have $|N(X)| > 2|X|$.

Proof: Suppose $|N(X)| \leq 2|X|$, for some $X \subseteq V(G)$, let $H = G[X \cup N(X)]$, so $V(H) \leq 3|X|$ and

$$e(H) \geq \frac{1}{2}\sum_{v\in X} d_G(v) \geq 3(d+1)|X| \geq (d+1)|V(H)|.$$

So H has a subgraph H' with $\delta(H') \geq d+1$. Apply moore bound to H',

$$3|X| \geq |V(H')| \geq 1 + (d+1)\sum_{i<g} d^i > d^g \Longrightarrow |X| > \frac{1}{3}d^g,$$

a contradiction.

Posa Lemma. If G is a praph $s.t.$ $|N(X)| > 2|X|$ for $\forall X \subseteq V(G)$ with $|X| \leq t$, then G contains a cycle of length at least $\min\{3t, n\}$ with a chord.

Lemma 4. If $G(L_i, L_{i+1})$ has a cycle C of length $2l$ with a chord, then for some $1 \leq m \leq i$, G has cycles

$$C_{2m+2}, C_{2m+4}, ..., C_{2m+2l-2}.$$

Proof of Theorem: Let G be a graph with $d(G) \geq 48(d+1)$ and girth $\geq 2g+1$, then G has a bipartite graph H with $d(H) \geq 24(d+1)$,

$$e(H) = \sum_{i\geq 0} e(L_i, L_{i+1}) \geq \frac{d(H)n}{2} = 12n(d+1),$$

then $\exists i, s.t.$

$$e(L_i, L_{i+1}) \geq 6(d+1)(|L_i| + |L_{i+1}|).$$

Assume not,

$$e(H) < 6(d+1)[(|L_0| + |L_1|) + (|L_1| + |L_2|) + ... + (|L_{t-1}| + |L_t|)]$$

for some t, it

$$= 6(d+1)(2n - |L_0| - |L_t|),$$

then $H(L_i, L_{i+1})$ has a subgraph H' with $\delta(H') \geq 6(d+1)$ and girth $\geq 2g+1$.
By above lemma, $\forall X \subseteq V(G)$ with $|X| \leq \frac{1}{3}d^g$, $|N(X)| > 2|X|$.
By Posa lemma, pick $t = \frac{1}{3}d^g$, so H' has a cycle of length $\geq d^g$ with a chord.
By lemma 4, G has $\Omega(d^g)$ cycles of consecutive even lengths.

Depentent Random Choice:

lemma 2. Let a, d, m, n, r be positive integers. Let G be a graph with $|V| = n$, and average $d = \frac{2|E(G)|}{n}$. If $\exists$ a positive integer t, $s.t.$

$$\frac{d^t}{n^{t-1}} - \binom{n}{r}(\frac{m}{n})^t \geq a,$$

then G contains a subset U of at least a vertices $s.t.$ every r vertices in U have at least m common neighbors.

Proof: Pick a set T of t vertices of V uniformly at random with repetition. Set $A = N(T)$, and let X denote the size of A, then the probability that $v \in V(G)$ is an element of A equals $(\frac{N(v)}{n})^t$. So by $Jensen'sInequality$, we have

$$E[X] = \sum_{v \in V(G)} (\frac{|N(v)|}{n})^t = n^{-t} \sum_{v \in V(G)} |N(v)|^2$$

$$\geq n^{1-t}(\frac{\sum_{v \in V(G)} |N(v)|}{n})^t = n^{1-t}(\frac{dn}{n})^t = \frac{d^t}{n^{t-1}}.$$

let Y be the random variable counting the number of subsets $S \subseteq A$ of size r with fewer than m common neighbors.
For a given S, the probability that it is a subset of A is $(\frac{|N(S)|}{n})^t$. So

$$E[Y] = \sum_S (\frac{|N(S)|}{n})^t \leq \binom{n}{r}(\frac{|N(S)|}{n})^t < \binom{n}{r}(\frac{m}{n})^t,$$

then

$$E[X-Y] \geq \frac{d^t}{n^{t-1}} - \binom{n}{r}(\frac{m}{n})^t \geq a.$$

Hence there $\exists$ a choice of T for with the conesponding set $A = N(T)$ satisfies $X - Y \geq a$. Delete one vertex from each subset S of A of size r with fewer than m common neighbors. Let U be the remaining subset, then $|U| \geq a$ and all subsets of size r have at least m common neighbors.

Theorem 2. If $H = (A \cup B, F)$ is a bipartite graph in which all vertices in B have degree $\leq r$, then

$$ex(n, H) \leq cH^{-\frac{1}{r}},$$

where $c = c(H)$ is only depends on H.

Proof: Using lemma 2:

Let $a = |A|, b = |B|, m = a + b, t = r$, and c=max$\{a^{\frac{1}{r}}, \frac{em}{r}\}$.

Suppose G with $v(G) = n$ and $e(G) > cn^{2-\frac{1}{r}}$, then the average degree $d \geq 2cn^{1-\frac{1}{r}}$.

$$\frac{d^t}{n^{t-1}} - \binom{n}{r}(\frac{m}{n})^t \geq \frac{(2cn^{1-\frac{1}{r}})^r}{n^{r-1}} - \frac{n^r}{r!}(\frac{m}{n})^r$$

$$= (2c)^r - \frac{m^r}{r!} \geq (2c)^r - (\frac{em}{r})^r.$$

(because $r! \geq (\frac{r}{e})^r \geq (2c)^r - c^r = (2^r - 1)c^r \geq a$)
Then we can find a vertex subset U of G with $|U| = a$, $s.t.$ all subsets of U of size r have $\geq m$ common neighbors.
Now we prove $H \subseteq G$:
We will find an embedding of H in G given by an injection

$$f : A \cup B \longrightarrow V(G).$$

Start by defining an injection

$$f : A \longrightarrow U$$

arbitrarily. Label the vertices of B as $v_1, v_2, ..., v_b$, we embed the vertices of B in this order one vertex at a time.
Suppose we need to embed $v_i \in B$, let $N_i \in A$ be those vertices of H adjacent to v_i, so $|N_i| \leq r$.
Since $f(N_i) \subseteq U$, and $|f(N_i)| \leq r$,there are $\geq a + b$ vertices adjacent to all vertices in $f(N_i)$. As the number of vertices already embedded $\leq a + b$. Then $\exists w \in V(G)$ which not be used and is adjacent to all vertices in $f(N_i)$ set $f(v_i) = w$. A contradiction!