Extremal and Probabilistic Graph Theory
Lecture 7
April 18th, Tuesday

## §1 Moore Graphs

- **Definition:** A walk $v_1 - v_2 - \cdots - v_k$ is *non-backtracking* if $v_{i+2} \neq v_i$ for $\forall\ i$.

- **Theorem 1(Alon-Hoory-Linial)** For a graph $G$ with average degree $d \geq 2$ and $n$ vertices, there are at least $nd(d-1)^{i-1}$ non-backtracking (oriented) walks of length $i \geq 1$ in $G$, with the equality if and only if $G$ is $d$-regular and $d$ is an integer.

- **Theorem 2(Moore bound)** Let $G$ have $n$ vertices, girth $\geq g$ and average degree $d \geq 2$. Then
$$n \geq n_0(g, d),$$
where
$$n_0(g,d) = \begin{cases} 1 + d\displaystyle\sum_{i=0}^{r-1}(d-1)^2, & g = 2r+1 \\ 2\displaystyle\sum_{i=0}^{r-1}(d-1)^i, & g = 2r \end{cases}$$
with equality if and only if $G$ is $d$-regular and of diameter $(\frac{g}{2}) + 1$.

  **Remark:** For these proofs, see the notes on April 21th, 2016.

- **Definition:** The graph which achieve this Moore bound are called *Moore graph.*

- **Theorem 3(Hoffimon-Singleton)** If a $d$-regular Moore graph of girth 5 exists, the $d \in \{2, 3, 7, 57\}$.

- **Theorem(Damerell)** For $d \geq 3$, if a $d$-regular Moore graph of girth $2g + 1$ exists, then $g \leq 2$.

- **Theorem(Feit-Higman)** For $g, d \geq 3$, if a $d$-regular Moore graph of girth $2g$ exists, then $g \in \{3, 4, 6\}$.

  We will show the existence of Moore graphs of girth $2g$ for $g \in \{3, 4, 6\}$.

- **Definition:** Given a hypergraph $H = (V, E)$, a *bigraph* $G$ of $H$ is a bipartite graph with parts $V$ and $E$, where $v \in V$ and $e \in E$ are adjacent in $G$ if and only if $v \in e$ in $H$.

- **Definition:** A *generalized $k$-gon of order $q$* is a $(q+1)$-uniform, $(q+1)$-regular hypergraph with $q^{k-1} + q^{k-2} + \cdots + q + 1$ vertices and No cycle of length at most $k - 1$.

- **Definition:** A *cycle* in hypergraphs of length $k$ means a collection of $k$ distinct hyperedges $e_1, e_2, \ldots, e_k$ and $k$ distinct vertices $v_1, v_2, \ldots, v_k$ such that $v_i \in e_i \cap e_{i+1}$ for $i \in \{1, 2, \cdots, k-1\}$ and $v_k \in e_k \cap e_1$, we also call it a *Berge cycle.*

- **Proposition 1:** The bigraph of a generalized $k$-gon od order $q$ is a $(q+1)$-regular Moore graph of girth $2k$.

  **Proof:** It follow by Theorem 2.

  So, next, we look for generalized $k$-gons for $k \in \{3, 4, 6\}$.

§2 **Projective graph**

- **Definition:** For a $n$-dimensional vector space $V$ over $\mathbb{F}_q$, let $\begin{bmatrix} V \\ k \end{bmatrix}_q$ denote the set of $k$-dimensional subspace of $V$.

  Define
$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q[n-1]_q \cdots [n-k+1]_q}{[k]_q[k-1]_q \cdots [1]_q},$$
  where $[i]_q = \frac{q^i-1}{q-1}$, called Guassion binomical coefficient.

- **Proposition 2**
$$\left| \begin{bmatrix} V \\ k \end{bmatrix}_q \right| = \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q^n-1)(q^{n-1}-1)\cdots(q^{n-k+1}-1)}{(q^k-1)(q^{k-1}-1)\cdots(q-1)}$$

  **Proof:** Note that
$$\left| \begin{bmatrix} V \\ k \end{bmatrix}_q \right| = \frac{(q^n-1)(q^n-q)\cdots(q^n-q^{k-1})}{(q^k-1)(q^k-q)\cdots(q^k-q^{k-1})}$$

- **Definition:** We call the element of $\begin{bmatrix} V \\ 1 \end{bmatrix}_q \begin{bmatrix} V \\ 2 \end{bmatrix}_q \begin{bmatrix} V \\ 3 \end{bmatrix}_q$ as *point*, *lines*, *planes*, respectively.

- **Definition:** For $n \geq k \geq l$, let $H_q[n, k, l]$ be the hypergraph with vertex set $\begin{bmatrix} V \\ l \end{bmatrix}_q$ and with edge set $\{ \begin{bmatrix} W \\ l \end{bmatrix}_q$, $W$ is a $k$-dim subgraph of $V$ $\}$. That is, each $k$-dim subgraph $W$ defines a hyperedge $\begin{bmatrix} W \\ l \end{bmatrix}_q$.

- **Proposition3:** $H_q[n, k, l]$ is a $\begin{bmatrix} k \\ l \end{bmatrix}_q$-uniform, $\begin{bmatrix} n-l \\ k-l \end{bmatrix}_q$-regular hypergraph on $\begin{bmatrix} n \\ l \end{bmatrix}_q$ vertices.

- **Definition:** The *$n$-dimensional projective space of order $q$*, denoted as $PG(n, q)$, is just $H_q[n+1, 2, 1]$.

- **Note:** $PG(n, q)$ has $1 + q + \cdots + q^n$ vertices. When $n = 2$, we call $PG(2, q)$ as *projective planes*

- **Theorem 5** The bigraph of $PG(2,q)$ is a $(q+1)$-regular Moore graph of girth 6.

  **Proof:** Exercise.

  Next, we consider Moore graph of girth 8, and we work in $PG(4,q)$.

- **Theorem 6(Benson)** Let $V$ be 5-dim space over $\mathbb{F}_q$. Let $\mathcal{S} = \{P \in \begin{bmatrix} V \\ 1 \end{bmatrix}_q ; P = span\{\vec{x}\}, \vec{x}\vec{x}^T = 0\}$. Let $\mathcal{L} = \{L \in \begin{bmatrix} V \\ 2 \end{bmatrix}_q ; L \subseteq \mathcal{S}\}$. Let $G$ be the bipartite graph with parts $\mathcal{S}$ and $\mathcal{L}$, where $P \in \mathcal{S}$ and $L \in \mathcal{L}$ are adjacent in $G$ if and only if $P \subseteq L$. Then $G$ is a $(q+1)$-regular Moore graph of girth 8.

  **Proof:** It is easy to see that for $\forall L \subseteq \mathcal{L}$, $d_G(L) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q = q+1$.

  **Claim 1:** $|\mathcal{S}| = \frac{q^4-1}{q-1} = q^3+q^2+q+1$.

  **Proof of claim 1:** For $P = span\{(x_1, \cdots, x_5)\} \in \mathcal{S}$, we have $x_1^2+x_2^2+x_3^2+x_4^2+x_5^2 = 0$. One can show: there are exactly $q^4-1$ non-zero vectors $(x_1, \cdots, x_5)$ satisfying this equation. Therefore, $|\mathcal{S}| = \frac{q^4-1}{q-1} = q^3+q^2+q+1$.

  **Claim 2:** $\forall P \in \mathcal{S}$, $d_G(P) = 1+q$.

  **Proof of claim 2:** Let $P = span\{\vec{x}\}$ for $\vec{x} = (x_1, \ldots, x_5)$, so we want how many $L \in \mathcal{L}$ such that $P \subseteq L$.
  Let us consider the properties on $L$.
  Suppose $L = span\{\vec{x}, \vec{y}\} \subseteq \mathcal{S}$. So,

$$\lambda\vec{x}+\mu\vec{y} \in \mathcal{S} \iff (\lambda\vec{x}+\mu\vec{y})(\lambda\vec{x}+\mu\vec{y})^T = 0 \ \& \ \vec{x}\vec{x}^T = 0 = \vec{y}\vec{y}^T \Rightarrow \vec{x}\vec{y}^T = 0.$$

  Therefore, $\vec{y}$ should satisfy: $\vec{y}\vec{y}^T = 0$ and $\vec{x}\vec{y}^T = 0$.

  There are exactly $q^3-q$ non-zero solutions $\vec{y} \notin span\{\vec{x}\}$. So there are exactly $\frac{q^3-q}{q-1}$ 1-dim subspaces $P' = span\{\vec{y}\}$. But there are exactly $q$ 1-dim subspaces $P'$, which plus $\vec{x}$ results in the same 2-dim $L \in \mathcal{L}$.

  Thus, there are exactly $1+q = \frac{q^3-q}{q(q-1)}$ many $L \in \mathcal{L}$ such that $P \subseteq L$.

  This proves claim 2.

  Claim 1&2 show that $G$ in $(q+1)$-regular and

$$|\mathcal{L}| = |\mathcal{S}| = 1+q+q^2+q^3.$$

  It remains to show $G$ is $C_4$-free and $C_6$-free.

  **Claim 3:** $G$ is $C_4$-free.

  **Proof of claim 3:** For any $L, L' \in \mathcal{L}$, $dim(L \cap L') \leq 1$, so there is at most 1 common neighbor of $L\&L'$, so $G$ is $C_4$-free.

  **Claim 4:** $G$ is $C_6$-free.

  **Proof of claim 4:** Suppose that $G$ has a $C_6$, say with distinct vertices $P_1, L_1, P_2, L_2, P_3, L_3$, so $P_1 \subseteq L_1 \cap L_3$, $P_2 \subseteq L_1 \cap L_2$ & $P_3 \subseteq L_2 \cap L_3$. Also, $P_i$ is self-orthogonal for $i \in [3]$, and any pair of $P_i$ and $P_j$ is also orthogonal.

Let $W = span\{P_1, P_2, P_3\}$, then $W \subseteq W^{\perp}$. But $dim(W) + dim(W^{\perp}) \leq dim(V) = 5$. Also, $dim(W) \leq dim(W^{\perp})$, so $dim(W) \leq 2$. So $P_i = P_j$, for some $i \neq j$. A contradiction.

This completes the proof of Theorem 6.