Extremal and Probabilistic Graph Theory
Lecture 8
April 20th, Thursday

Recall that:

- **Theorem 6(Benson)** Let V be a 5-dim space over $\mathbb{F}_q$. Let $A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$, and $\mathcal{S} = \{P \in \begin{bmatrix} V \\ 1 \end{bmatrix}_q ; P = span\{\vec{x}\},\ \vec{x}A\vec{x}^T = 0\}$, Let $\mathcal{L} = \{L \in \begin{bmatrix} V \\ 2 \end{bmatrix}_q ; L \subseteq \mathcal{S}\}$. Let G be the bipartite graph with parts $\mathcal{S}$ and $\mathcal{L}$, where $P \in \mathcal{S}$ and $L \in \mathcal{L}$ are adjacent in G if and only if $P \subseteq L$. Then G is a $(q+1)$-regular Moore graph of girth 8.

  Remark: Replace A by any non-degenerated 5×5 matrix. Then the same construction will also give a $(q+1)$-regular Moore graph of girth 8.

- **Theorem 7** Let $k \in \{2, 3, 5\}$ and q be a prime power. Then there exists $(q+1)$-regular Moore graph of girth $2k+2$.

- **Definition:** Let $\mathcal{C}_k = \{C_3, C_4, \cdots, C_k\}$. Let the *Zarankiewicz number* $Z(m, n, F)$ be the maximum number of edges in an $m \times n$-bipartite F-free graph.

- **Theorem 8** Let $k \in \{2, 3, 5\}$. If $n = q^k + q^{k-1} + \cdots + q + 1$ for some prime power q, then $Z(n, n, \mathcal{C}_{2k}) = (q+1)n$. For general n, $Z(n, n, \mathcal{C}_{2k}) = (1 + o(1))n^{1+\frac{1}{k}}$.

  Proof: Let x be the unique real such that $n = x^k + x^{k-1} + \cdots + x + 1$.

  Upper bound: Let G be an $n \times n$-bipartite graph of girth$\geq 2k+2$, and average degree $d + 1 \geq 2$.

  By Theorem 1, G has $\geq 2n(d+1)d^{i-1}$ non-backtracking (oriented) walks of length $i \geq 1$. So G has $\geq 2n(d+1)\sum_{i=1}^{k+1} d^{i-1}$ non-backtracking oriented walks of length at most $k+1$.

  Note that $e(G) = n(d+1)$. Thus, by averaging there is an edge $uv \in E(G)$ such that the number of non-backtracking walks of length $\leq k+1$ with uv as the leading edge is at least $\frac{2n(d+1)\sum_{i=1}^{k+1} d^{i-1}}{e(G)} \geq 2\sum_{i=1}^{k+1} d^{i-1} = 2(1 + d + \cdots + d^k)$. Since girth $\geq 2k+2$, the ends of all those walks with leading edge uv are distinct implying that

  $$2n = |V(G)| \geq \#\text{walks of length at most } k+1 \text{ with leading edge } uv \geq 2(1 + d + \cdots + d^k).$$

  So,
  $$1 + x + x^2 + \cdots + x^k = n \geq 1 + d + d^2 + \cdots + d^k.$$

  So, $x \geq d$, and thus $e(G) = n(d+1) \leq (1+x)n$, where $n = 1 + x + x^2 + \cdots + x^k$.

  For $n = 1 + q + \cdots + q^k$, then $e(G) \leq (1+q)n$.

  For general n, we have $x = (1 + o(1))n^{\frac{1}{k}}$, so $e(G) \leq (1 + o(1))n^{1+\frac{1}{k}}$.

  Lower bound: By the Theorem 7, $\exists(1+q)$-regular Moore graph of girth $2k+2$, which are bipartite and have exactly $2(1 + q + \cdots + q^k)$ vertices.

So, this shows that for $n = 1 + q + \cdots + q^k$, then $Z(n, n, \mathcal{C}_{2k}) = (q+1)n$.

For general n, lat q be the maximal prime such that $n_q \triangleq 1 + q + \cdots + q^k \leq n$. Since $\lim_{r \to +\infty} \frac{p_r}{p_{r+1}} = 1$, where $p_r = r^{th}$ prime, we have that $q \backsim n^{\frac{1}{k}}$.

So,

$$Z(n, n, \mathcal{C}_{2k}) \geq Z(n_q, n_q, \mathcal{C}_{2k}) = (1+q)n_q = (1 + o(1))n^{1+\frac{1}{k}}.$$

Theorem 8 is done.

- **Corollary:**

$$ex(n, C_4) \geq ex(n, \mathcal{C}_4) \geq Z(\frac{n}{2}, \frac{n}{2}, \mathcal{C}_4) \geq (1 + o(1))(\frac{n}{2})^{\frac{3}{2}},$$

and

$$ex(n, C_6) \geq (1 + o(1))(\frac{n}{2})^{\frac{4}{3}}.$$

- **Theorem 9(Reiman)**

$$ex(n, C_4) \leq \frac{n}{4}(1 + \sqrt{4n-3}) \approx \frac{1}{2}n^{\frac{3}{2}}$$

- **Theorem 10**

$$ex(n, C_4) = \frac{1}{2}n^{\frac{3}{2}} + o(n^{\frac{3}{2}}).$$

Proof: We define the *Erdös-Renyi polarity graph* ER_q, that is the graph with vertex-set $\begin{bmatrix} V \\ 1 \end{bmatrix}_q$ (Where V is a 3-dim space over $\mathbb{F}_q$), where $U, W \in \begin{bmatrix} V \\ 1 \end{bmatrix}_q$ are adjacent if and only if $U \perp W$.

In other words, if $U = Span\{(x, y, z)\}$, and $W = Span\{(u, v, w)\}$, then $xu + yv + zw = 0$. It is easy to check:

1. $|V(ER_q)| = \begin{bmatrix} 3 \\ 1 \end{bmatrix}_q = 1 + q + q^2$.
2. For $\forall\, U = span\{(x, y, z)\} \in \begin{bmatrix} V \\ 1 \end{bmatrix}_q$, there are exactly q^2-1 non-zero solutions (u, v, w) to be equation $xu + yv + zw = 0$.

So, there are exactly $\frac{q^2-1}{q-1} = 1 + q$ 1-dim subspace $W \in \begin{bmatrix} V \\ 1 \end{bmatrix}_q$ perpendicular to U.

1. # 1-dim self-orthogonal subspaces is $1 + q$, so there vertices (self-orthogonal) have degree q, and other vertices have degree $1 + q$.

$$\Rightarrow e(ER_q) = \frac{1}{2}(1+q)q + \frac{1}{2}q^2(1+q) = \frac{1}{2}q(1+q)^2.$$

2. ER_q is C_4-free, as for fixed (x_1, y_1, z_1) & (x_2, y_2, z_2), then there is exactly 1 solution to $x_iu + y_iv + z_iw = 0$ for $i = 1, 2$.

Together, ER_q is C_4-free and has $1+q+q^2$ vertices and $\frac{1}{2}q(1+q)^2$ edges.

$$\Rightarrow ex(n, C_4) \geq (1+o(1))\frac{1}{2}n^{\frac{3}{2}}.$$

Remark: ER_q contains $\frac{1}{6}q^3 + o(q^2)$ triangles, as each edge is contained in a unique triangle.

- **Theorem 11(Furedi)** Let G be an extremal C_4-free graph with $n = 1+q+q^2$ vertices. Then, $e(G) \leq \frac{1}{2}q(1+q)^2$, with equality if and only if $G = ER_q$.

§3 **Polarity graph**

Let $\mathcal{P}$ and $\mathcal{L}$ be distinct sets, the elements of which are called *points* and *lines*, respectively.

A subset $I \subseteq \mathcal{P} \times \mathcal{L}$ is called *incidence relation* on the pair $(\mathcal{P}, \mathcal{L})$; and the triple $(\mathcal{P}, \mathcal{L}, I)$ is called a *rank two geometry*.

- **Definition:** The *incidence graph* G of $(\mathcal{P}, \mathcal{L}, I)$ is a bipartite graph on parts $\mathcal{P}$ and $\mathcal{L}$, where $P \in \mathcal{P}$ and $L \in \mathcal{L}$ are adjacent if and only if $(P, L) \in I$.

- **Definition:** A *polarity* of $(\mathcal{P}, \mathcal{L}, I)$ is a bijection π: $\mathcal{P} \cup \mathcal{L} \to \mathcal{P} \cup \mathcal{L}$ such that:

  (1) $\mathcal{P}^\pi = \mathcal{L}$ & $\mathcal{L}^\pi = \mathcal{P}$,

  (2) $\forall\ P \in \mathcal{P}$, $L \in \mathcal{L}$, we have $(P, L) \in I \iff (L^\pi, P^\pi) \in I$,

  (3) $\pi^2 = 1$.

- **Definition:** For a polarity π of $(\mathcal{P}, \mathcal{L}, I)$, the *polarity graph* G^π is a graph with vertex-set $\mathcal{P}$ and edge-set $E(G^\pi) = \{P_1P_2 |\ P_1 \neq P_2 \in \mathcal{P},\ (P_1, P_2^\pi) \in I\}$.

- **Definition:** We say a point $P \in \mathcal{P}$ is an *absolute point*, if $(P, P^\pi) \in I$. Let $N_\pi =$ #absolute points (w.r.t. π).

- **Theorem 12** Let π be a polarity of $(\mathcal{P}, \mathcal{L}, I)$. Then,

  (i) $deg_{G^\pi}(P) = deg_G(P) = 1$, if P is an absolute point and $deg_{G^\pi}(P) = deg_G(P)$, otherwise;

  (ii) $|V(G^\pi)| = \frac{1}{2}|V(G)|$ and $|E(G^\pi)| = |E(G)| - N_\pi$;

  (iii) If G^π contains a C_{2k+1}, then G contains a C_{4k+2};

  (iv) If G^π contains a C_{2k}, then G contains two vertex-disjoint $2k$-cycle C and C' such that $C^\pi = C'$. In particular, if G is C_{2k}-free, then G^π is C_{2k}-free.

  (v) $g(G^\pi) \geq \frac{1}{2}g(G)$.