Extremal and Probabilistic Graph Theory
Lecture 9
April 25th, Tuesday

Recall that:

- **Theorem 10** $ex(n, C_4) = (1 + o(1))\frac{1}{2}n^{\frac{3}{2}}$.

  Proof: The upper bound is from *Rieman's bound.*

  The lower bound is from the construction of *Erdös-Reny polarity graph* ER_q, where $V(ER_q) = \begin{bmatrix} V \\ 1 \end{bmatrix}_q$ (V is a 3-dim space), and $U, W \in \begin{bmatrix} V \\ 1 \end{bmatrix}_q$ adjacent if and only if $U \perp W$. i.e. $U = Span\{(x_1, x_2, x_3)\}$, and $W = Span\{(y_1, y_2, y_3)\}$, then $x_1y_1 + x_2y_2 + x_3y_3 = 0$ over $\mathbb{F}_q$. So $1+q$ vertices of degree q & q^2 vertices of degree $1+q$, $\Rightarrow e(ER_q) = \frac{1}{2}q(1+q)^2 \approx \frac{1}{2}n^{\frac{3}{2}}$ & $v(ER_q) = 1 + q + q^2 = n$.

  Then,

  1. ER_q contains triangles.
  2. Moore graph of girth 6 $\Rightarrow ex(n, \{C_3, C_4, C_5\} \geq (\frac{n}{2})^{\frac{3}{2}}$.
  3. Recall the general definition of polarity graphs.

- **Theorem 13(F*ü*redi-Naor-Verstrate)**

$$0.538n^{\frac{4}{3}} \leq ex(n, C_6) \leq 0.627n^{\frac{4}{3}}.$$

$$0.58n^{\frac{6}{5}} \approx \frac{4}{5^{6/5}}n^{\frac{6}{5}} \leq ex(n, C_{10}) \leq O(n^{\frac{6}{5}}).$$

  These disprove a conjecture of Erdös-Simonovits that $ex(n, C_{2k}) = \frac{1+o(1)}{2}n^{1+\frac{1}{k}}$, $\forall k \geq 2$.

  As we recall,

- **Theorem 14**

$$(\frac{n}{2})^{\frac{3}{2}} + o(n^{\frac{3}{2}}) \leq ex(n, \mathcal{C}_4) \leq \frac{n^{\frac{3}{2}}}{2} + o(n^{\frac{3}{2}}).$$

- **Conjecture(Erd*ö*s)**

$$ex(n, \mathcal{C}_4) = (\frac{n}{2})^{\frac{3}{2}} + o(n^{\frac{3}{2}}).$$

  Where $\mathcal{C}_4 = \{C_3, C_4\}$.

§ 4 Hypergraphs

- **Definition:** A *Berge k-cycle* in r-graphs is a r-graph consisting of distinct hyperedges $e_1, e_2, \cdots, e_k$ such that there exist k distinct vertices $v_1, v_2, \cdots, v_k$ such that $v_i \in e_i \cap e_{i+1}$ for $1 \leq i \leq k-1$ and $v_k \in e_k \cap e_1$.

- **Definition:** For an integer g, let $\mathcal{C}_g = \{C_2, C_3, \cdots, C_g\}$.

  Similarly, let $ex_r(n, \mathcal{C}_g)$ be the *Turán number*, i.e. the maximum number of edges in an n-vertex $\mathcal{C}_g$-free r-graphs.

  Fact:

$$ex_r(n, \mathcal{C}_2) = \frac{\binom{n}{2}}{\binom{r}{2}} + o(n^2).$$

- **Theorem 15 (Ruzsa-Szemerédi (6,3)-Theorem)**

$$\Omega(n^{2-\varepsilon}) = 2^{-c\sqrt{\log n}} n^2 \leq ex_r(n, \mathcal{C}_3) \leq o(n^2).$$

- **Theorem(Lazebrok-Verstrete)**

  (1) $ex_3(n, \mathcal{C}_4) \leq \frac{1}{6} n \sqrt{n - \frac{3}{4}} + \frac{n}{12}$.

  (2) There exists an 3-graph H on $n = q^3$ vertices, of girth 5 and with $e(H) = \binom{q+1}{3} = \frac{1}{6} n^{\frac{3}{2}} - \frac{1}{6}\sqrt{n}$.

  Therefore, $ex_3(n, \mathcal{C}_4) = (\frac{1}{6} + o(1)) n^{\frac{3}{2}}$.

  Proof: **Upper Bound:** We prove a stronger result:

- **Theorem** Let H be an r-graph on n vertices and of girth ≥ 5, then

$$e(H) \leq \frac{1}{r(r-1)} n^{\frac{3}{2}} + \frac{r-2}{2r(r-1)} n + O(\sqrt{n}).$$

  Proof: Let $m = |E(H)|$, for a fixed $v \in V(H)$ and for any unordered pair of edges A,B that contain v, let $v(A, B)$ be the set of unordered pair (a, b) of vertices, where $a \in A - v$, $b \in B - v$.

  Note that for any $A, B \neq C, D$, $v(A, B) \cap v(C, D) = \phi$.

  Define

$$D_v = \bigcup_{\{A,B\}, v \in A \cap B} v(A, B).$$

  So $|D_v| = \binom{d_v}{2}(r-1)^2$, where d_v denotes the degree of v in H.

  Claim 1: For $u \neq v$, $D_u \cap D_v = \phi$.(Otherwise $\exists C_4$.)

  Claim 2: No pairs in D_v is contained in an edge.(Otherwise $\exists C_3$.)

  Since H contains No 2-cycles, and the number of pairs of vertices contained in edges is exactly $\binom{r}{2} m$, by claim 2,

$$\begin{aligned}\binom{n}{2} - \binom{r}{2} m &\geq \#\{a, u, b\} \text{ where } au,\ bu \text{ are in different edges} \\ &= \sum_v |D_v| \\ &= (r-1)^2 \sum_v \binom{d_v}{2} \\ &\geq (r-1)^2 (\frac{r^2 m^2}{2n} - \frac{rm}{2}).\end{aligned}$$

as $\sum_v d_v = rm$.

$$\Rightarrow r^2(r-1)^2m^2 - r(r-1)(r-2)nm - n^2(n-1) \leq 0$$

$$\Rightarrow e(H) = m \leq \frac{n}{r(r-1)}\sqrt{n + \frac{r^2-4r}{4}} + \frac{r-2}{2r(r-1)}n.$$

Lower bound:

Consider ER_q, we say a vertex $U \in \begin{bmatrix} V \\ 1 \end{bmatrix}_q$ is *isotropic* if $U \perp U$.

Recall a *non-degenerate* orthogonal geometry on $V = \mathbb{F}_q^3$ corresponding to the bilinear form $\vec{x}\vec{y} = x_1y_1 + x_2y_2 + x_3y_3$.

The non-degenerate means that No non-zero vertex of V is orthogonal to all vertices of V.

Claim 1: Any isotropic vertex is not contained in triangles.

Proof: $\vec{x} \cdot \vec{x} = 0$, $\vec{x} \perp span\{\vec{x}, \vec{y}, \vec{z}\} = V$, so $\vec{x} = 0$. A contradiction.

Claim 2: No edges of ER_q which has 2 isotropic vertices.

Proof: Suppose not, $\vec{x} \cdot \vec{x} = 0$, $\vec{y} \cdot \vec{y} = 0$ & $\vec{x} \cdot \vec{y} = 0$, assume $span\{\vec{x}\} \neq span\{\vec{y}\}$.

Consider the 2-dim subspace $W = span\{\vec{x}, \vec{y}\}$. $\Rightarrow \vec{x} \perp W$, $\vec{y} \perp W$. Since the geometry is non-degenerate, the orthogonal complement of 2-dim subspace is a 1-dim subspace. So, $span\{\vec{x}\} = span\{\vec{y}\}$, a contradiction.

Let H be a 3-graph obtained from the set of non-isotropic of ER_q, such that $\{U_1, U_2, U_3\} \in E(H)$ if and only if $U_1U_2U_3$ is a triangle of ER_q.

It is easy to see that H is $\mathcal{C}_4$-free.

$$\#\text{edges in the induced subgraph of } ER_q \text{ on the non-isotropic vertices}$$

$$= e(ER_q) - q(q+1) = \frac{(q+1)q(q-1)}{2}.$$

Since each such edge is in one triangle, we get that,

$$e(H) = \#\text{triangles in } ER_q = \frac{(1+q)q(q-1)}{6} = \binom{q+1}{3}.$$

And H contains 5-cycle ($q \geq 27$).