Extremal and Probabilistic Graph Theory
April 27th, Thursday

Projective $k_{3,3}$-free norm graphs

Definition: The graph $H = H(q,3)$ is defined as follows:
Let $V(H) = \mathbb{F}_{q^2} \times \mathbb{F}_q^*$.
Two distinct vertices (A,a) and (B,b) are adjacent if and only if $N(A+B) = ab$, where $N(X) = X^{1+q}$ for $X \in \mathbb{F}_{q^2}$.

Remark: Norm of $\mathbb{F}_{q^l}$: $N_l(X) = X \cdot X^q \cdot \dots \cdot X^{q^{l-1}}$ for $X \in \mathbb{F}_{q^l}$
$(N_l(X))^q = N_l(X)$

So we have $|V(H)| = q^3 - q^2$.
If (A,a) and (B,b) are adjacent, then (A,a) and $B \neq -A$ determine b. So the degree of each vertex in $V(H)$ is either q^2-1 or q^2-2 (when $N(A+A) = a^2$).

Theorem 1: The graph $H = H(q,3)$ contains no subgraph isomorphic to $K_{3,3}$. Thus there exists a constant C such that for every $n = q^3 - q^2$, where q is a prime power, $ex(n, K_{3,3}) \geqslant \frac{1}{2}n^{\frac{5}{3}} + \frac{1}{3}n^{\frac{4}{3}} + C$.

Remark: The upper bound of Füredi is $ex(n, K_{3,3}) \leqslant \frac{1}{2}n^{\frac{5}{3}} + n^{\frac{4}{3}} + 3n$. $(\star)$

Lemma 1: Let $\mathbb{K}$ be a field, and $a_{ij}, b_i \in \mathbb{K}$ for $1 \leqslant i,j \leqslant 2$ such that $a_{1j} \neq a_{2j}$. Then the system of equations

$$(x_1 - a_{11})(x_2 - a_{12}) = b_1 \tag{1}$$

$$(x_2 - a_{21})(x_2 - a_{22}) = b_2 \tag{2}$$

has at most 2 solutions $(x_1, x_2) \in \mathbb{K}^2$.

Proof: (2)$-$(1), we get $(a_{11} - a_{21})x_2 + (a_{12} - a_{22})x_1 + a_{21}a_{22} - a_{11}a_{22} = b_2 - b_1$.
Hence we can express x_1 in terms of a linear function of x_2.
Substituting this back to (1), we obtain a quadratic equation in x_2 with a nonzero leading coefficient.
This equation has at most 2 solutions in x_2, and each one determines x_1 uniquely. □

Lemma 2: If $(D_1,d_1),(D_2,d_2),(D_3,d_3)$ are distinct elements of $V(H)$, then the system of equations

$$N(X + D_1) = xd_1 \tag{4}$$

$$N(X + D_2) = xd_2 \tag{5}$$

$$N(X + D_3) = xd_3 \tag{6}$$

has at most 2 solutions $(X,x) \in \mathbb{F}_{q^2} \times \mathbb{F}_q^*$.

Proof: Suppose $\exists$ a solution (X,x), then $x \neq -D_i$ for $i=1,2,3$ and $D_i \neq D_j$ for $i \neq j$ (Since if $D_i = D_j$, we have $d_i = d_j$).

$\Rightarrow N(X+D_3) \in \mathbb{F}_q^*,\ N(D_i - D_2) \in \mathbb{F}_q^*\ (i=1,2)$

$\frac{(4)\times(5)}{(6)} \Rightarrow N((X+D_i)/(X+D_3)) = d_i/d_3\ (i=1,2)$

Divide each of the above equation by $N(D_i - D_3)$, we get $N(Y+A_i) = (Y+A_i)(Y^q + A_i^q) = b_i$, where $Y = 1/(X+D_3)$, $A_i = 1/(D_i - D_3)$ and $b_i = d_i/(d_3 N(D_i - D_3))$ for $i = 1,2$. (Using $(A+B)^q = A^q + B^q$ for all A, B in $\mathbb{F}_{q^2}$)

By **Lemma 1**, it has at most 2 solutions. □

Ramsey Numbers

Definition: The k-color Ramsey number $R_k(G)$ is the maximum integer m such that one can color the edges of the complete graph K_m using k colors with no monochromatic copy of G.

The multicolor Ramsey number of a bipartite graph is strongly related to its Turán number:

On one hand, we have $k \cdot ex(R_k(G), G) \geqslant \binom{R_k(G)}{2}$ $(\star\star)$. Hence an upper bound on the Turán number can immediately be converted into an upper bound on the multicolor Ramsey number.

On the other hand, we obtain a lower bound for the Ramsey number from a lower bound on the Turán number if our construction of a G-free graph can be used to construct an (almost) complete tiling of the complete graph.

Theorem (Chung, Graham, Spencer): $ck^3/log^3 k \leqslant R_k(K_{3,3}) \leqslant (2+o(1))k^3$

We will use $H(q,3)$ to obtain an asymptotic formula for $R_k(K_{3,3})$.

Fact: There is a prime number between n and $n + o(n)$.

Theorem 2: $R_k(K_{3,3}) = (1+o(1))k^3$

Proof: Knowing Füredi's upper bound $(\star)$ for the Turán number of $K_{3,3}$-free, inequality $(\star\star)$ provides $R_k(G) \leqslant (1+o(1))k^3$.

For the other direction, we define an almost complete $(q-1)$-coloring of the edges of $K_{(q^2-1)(q-1)}$ such that there is no monochromatic $K_{3,3}$. The edges that are missing from disjoint complete bipartite graphs of order $2q-2$ and thus can be colored recursively.

The vertices are labeled by the elements in $\mathbb{F}_{q^2} \times \mathbb{F}_q^*$. If $A \neq -B$, color the edge between (A,a) and (B,b) by $N(A+B)/ab$. This way, no color class contains a $K_{3,3}$. The proof of **Theorem 1** works for any fixed color because of the generality of **Lemma 1**.

The uncolored edges form $(q^2-1)/2$ pairwise disjoint complete bipartite graphs, each of which has $2(q-1)$ vertices. Using the same construction recursively, one can color the edges of each such bipartite graph using at most $(1+o(1))(2q)^{\frac{1}{3}}$ additional colors. (Since the uncolored copies of the graphs $K_{q-1,q-1}$ are pairwise disjoint, we can use the same set of new colors for each of them.)

The total number of colors is thus $q+o(q)$, implying the lower bound by the **Fact**. □

General norm-graphs (Koll*á*, R*ó*nyai, and Szab*ó*)

First, we give (but not prove) a generalization of **Lemma 1**.

Lemma 3: Let $\mathbb{K}$ be a field, and $a_{ij}, b_i \in \mathbb{K}$ for $1 \leqslant i, j \leqslant t$ such that $a_{i_1 j} \neq a_{i_2 j}$. Then the system of equations

$$\begin{aligned}
(x_1-a_{11})(x_2-a_{12})\cdots(x_t-a_{1t}) &= b_1\\
(x_1-a_{21})(x_2-a_{22})\cdots(x_t-a_{2t}) &= b_2\\
&\vdots\\
(x_1-a_{t1})(x_2-a_{t2})\cdots(x_t-a_{tt}) &= b_t
\end{aligned}$$

has at most $t!$ solutions $(x_1,\cdots,x_t)\in\mathbb{K}^t$.

Definition (the general projective norm-graphs): We define the norm graph $G=G_{q,t}$ as follows:

Let $V(G)=\mathbb{F}_{q^t}$.

For $a\in\mathbb{F}_{q^t}$, let $N(a)$ denote the $\mathbb{F}_{q^t}/\mathbb{F}_q$-norm of a. i.e. $N(a)=a\cdot a^q\cdot\cdots\cdot a^{q^{t-1}}=a^{(q^t-1)/(q-1)}\in\mathbb{F}_q$.

Let $a\neq b\in V(G)$ be adjacent if and only if $N(a+b)=1$.

Fact: The number of solutions in $\mathbb{F}_{q^t}$ of the equation $N(a)=1$ is $(q^t-1)/(q-1)$.

Thus, let $n=q^t=v(G)$, then $e(G)\geqslant\frac{1}{2}q^t(\frac{q^t-1}{q-1}-1)\geqslant\frac{1}{2}q^{2t-1}=\frac{1}{2}n^{2-\frac{1}{t}}$.

Theorem 3: The graph $G=G(q,t)$ contains no subgraph isomorphic to $K_{t,t!+1}$.

Corollary 1: For $t\geqslant 2$ and $s\geqslant t!+1$, we have $ex(n,K_{t,s})\geqslant c_t\cdot n^{2-\frac{1}{t}}$, where $c_t>0$ is a constant depending on t, we may choose $c_t=2^{-t}$. For every t and $s\geqslant t$, the inequality holds with $c=\frac{1}{2}$ for infinitely many values of n.

Proof of Theorem 3: If $d_1,\cdots,d_t$ are t distinct elements from $\mathbb{F}_{q^t}$, and we choose $\mathbb{K}=\mathbb{F}_{q^t}$, $a_{ij}=-d_j^{q^{i-1}}$, $x_j=x^{q^{j-1}}$, $b_j=1$ in **Lemma 3**, then the system of equations

$$N(x+d_1)+(x+d_1)(x^q+d_1^q)\cdots(x^{q^{t-1}}+d_1^{q^{t-1}})=1$$
$$\vdots$$
$$N(x+d_t)+(x+d_t)(x^q+d_t^q)\cdots(x^{q^{t-1}}+d_t^{q^{t-1}})=1$$

has at most $t!$ solutions $x \in \mathbb{F}_{q^t}$. □

Then, with these techniques, Alon *et al.* gave an improved construction.

Definition(the general projective norm-graphs): For any $t > 2$, define $H = H_{q,t}$ as follows:

Let $V(H) = \mathbb{F}_{q^{t-1}} \times \mathbb{F}_q^*$.

Two distinct (A, a) and $(B, b) \in V(H)$ are adjacent if and only if $N(A+B) = ab$, where $N(X) = X^{1+q+\cdots+q^{t-2}}$.

Note that: (1) $|V(H)| = q^t - q^{t-1}$; (2) If (A, a) and (B, b) are adjacent, then (A, a) and $B \neq -A$ determine b.