Projective $k_{3,3}$-free norm graphs

**Definition:** The graph $H = H(q, 3)$ is defined as follows:

Let $V(H) = \mathbb{F}_q^2 \times \mathbb{F}_q^*$. Two distinct vertices $(A, a)$ and $(B, b)$ are adjacent if and only if $N(A+B) = ab$, where $N(X) = X^{1+q}$ for $X \in \mathbb{F}_q^2$.

**Remark:** Norm of $\mathbb{F}_q^l$: $N_l(X) = X \cdot X^q \cdots X^{q^{l-1}}$ for $X \in \mathbb{F}_q^l$.

$N_l(X)^9 = N_l(X)$

So we have $|V(H)| = q^3 - q^2$.

If $(A, a)$ and $(B, b)$ are adjacent, then $(A, a)$ and $B \neq -A$ determine $b$. So the degree of each vertex in $V(H)$ is either $q^2 - 1$ or $q^2 - 2$ (when $N(A+A) = a^2$).

**Theorem 1:** The graph $H = H(q, 3)$ contains no subgraph isomorphic to $K_{3,3}$. Thus there exists a constant $C$ such that for every $n = q^3 - q^2$, where $q$ is a prime power, $ex(n, K_{3,3}) \geq \frac{1}{2} n^{5/3} + \frac{1}{3} n^{4/3} + C$.

**Remark:** The upper bound of F"{u}redi is $ex(n, K_{3,3}) \leq \frac{1}{2} n^{5/3} + n^{2/3} + 3n$. ($\star$)

**Lemma 1:** Let $\mathbb{K}$ be a field, and $a_{ij}, b_i \in \mathbb{K}$ for $1 \leq i, j \leq 2$ such that $a_{1j} \neq a_{2j}$. Then the system of equations

1. $(x_1 - a_{11})(x_2 - a_{12}) = b_1$ \hspace{1cm} (1)
2. $(x_2 - a_{21})(x_2 - a_{22}) = b_2$ \hspace{1cm} (2)

has at most 2 solutions $(x_1, x_2) \in \mathbb{K}^2$.

**Proof:** (2)−(1), we get $(a_{11} - a_{21})x_2 + (a_{12} - a_{22})x_1 + a_{21}a_{22} - a_{11}a_{22} = b_2 - b_1$.

Hence we can express $x_1$ in terms of a linear function of $x_2$.

Substituting this back to (1), we obtain a quadratic equation in $x_2$ with a nonzero leading coefficient.

This equation has at most 2 solutions in $x_2$, and each one determines $x_1$ uniquely. $\square$

**Lemma 2:** If $(D_1, d_1), (D_2, d_2), (D_3, d_3)$ are distinct elements of $V(H)$, then the system of equations

1. $N(X + D_1) = xd_1$ \hspace{1cm} (4)
2. $N(X + D_2) = xd_2$ \hspace{1cm} (5)
3. $N(X + D_3) = xd_3$ \hspace{1cm} (6)
has at most 2 solutions \((X, x) \in \mathbb{F}_q^2 \times \mathbb{F}_q^*\).

**Proof:** Suppose \(\exists\) a solution \((X, x)\), then \(x \neq -D_i\) for \(i = 1, 2, 3\) and \(D_i \neq D_j\) for \(i \neq j\) (Since if \(D_i = D_j\), we have \(d_i = d_j\)).

\[ N(X + D_3) \in \mathbb{F}_q^*, \quad N(D_i - D_2) \in \mathbb{F}_q^* (i = 1, 2) \]

Divide each of the above equation by \(N(D_i - D_3)\), we get \(N(Y + A_i) = (Y + A_i)(Y_q^q + A_i^q) = b_i\), where \(Y = 1/(X + D_3)\), \(A_i = 1/(D_i - D_3)\) and \(b_i = d_i/(d_3N(D_i - D_3))\) for \(i = 1, 2\). (Using \((A + B)^q = A^q + B^q\) for all \(A, B\) in \(\mathbb{F}_q^q\))

By Lemma 1, it has at most 2 solutions. \(\square\)

## Ramsey Numbers

**Definition:** The \(k\)-color Ramsey number \(R_k(G)\) is the maximum integer \(m\) such that one can color the edges of the complete graph \(K_m\) using \(k\) colors with no monochromatic copy of \(G\).

The multicolor Ramsey number of a bipartite graph is strongly related to its Turán number:

On one hand, we have \(k \cdot \text{ex}(R_k(G), G) \geq \left(\frac{R_k(G)}{2}\right)\) (**)\). Hence an upper bound on the Turán number can immediately be converted into an upper bound on the multicolor Ramsey number.

On the other hand, we obtain a lower bound for the Ramsey number from a lower bound on the Turán number if our construction of a \(G\)-free graph can be used to construct an (almost) complete tiling of the complete graph.

**Theorem** (Chung, Graham, Spencer): \(ck^3/\log^3k \leq R_k(K_{3,3}) \leq (2 + o(1))k^3\)

We will use \(H(q, 3)\) to obtain an asymptotic formula for \(R_k(K_{3,3})\).

**Fact:** There is a prime number between \(n\) and \(n + o(n)\).

**Theorem 2:** \(R_k(K_{3,3}) = (1 + o(1))k^3\)

**Proof:** Knowing Füredi’s upper bound \((\ast)\) for the Turán number of \(K_{3,3}\)-free, inequality (**\)) provides \(R_k(G) \leq (1 + o(1))k^3\).

For the other direction, we define an almost complete \((q - 1)\)-coloring of the edges of \(K_{(q^2-1)/(q-1)}\) such that there is no monochromatic \(K_{3,3}\). The edges that are missing from disjoint complete bipartite graphs of order \(2q - 2\) and thus can be colored recursively.

The vertices are labeled by the elements in \(\mathbb{F}_{q^2} \times \mathbb{F}_q^*\). If \(A \neq -B\), color the edge between \((A, a)\) and \((B, b)\) by \(N(A + B)/ab\). This way, no color class contains a \(K_{3,3}\). The proof of **Theorem 1** works for any fixed color because of
the generality of Lemma 1.

The uncolored edges form \((q^2 - 1)/2\) pairwise disjoint complete bipartite graphs, each of which has \(2(q - 1)\) vertices. Using the same construction recursively, one can color the edges of each such bipartite graph using at most \((1 + o(1))(2q)\alpha\) additional colors. (Since the uncolored copies of the graphs \(K_{q-1,q-1}\) are pairwise disjoint, we can use the same set of new colors for each of them.)

The total number of colors is thus \(q + o(q)\), implying the lower bound by the Fact.

General norm-graphs (Kollá, Rónyai, and Szabó)

First, we give (but not prove) a generalization of Lemma 1.

Lemma 3: Let \(K\) be a field, and \(a_{ij}, b_i \in K\) for \(1 \leq i, j \leq t\) such that \(a_{i1} \neq a_{i2}\). Then the system of equations

\[
\begin{align*}
(x_1 - a_{11})(x_2 - a_{12}) \cdots (x_t - a_{1t}) &= b_1 \\
(x_1 - a_{21})(x_2 - a_{22}) \cdots (x_t - a_{2t}) &= b_2 \\
& \vdots \\
(x_1 - a_{t1})(x_2 - a_{t2}) \cdots (x_t - a_{tt}) &= b_t
\end{align*}
\]

has at most \(t!\) solutions \((x_1, \ldots, x_t) \in K^t\).

Definition (the general projective norm-graphs): We define the norm graph \(G = G_{q,t}\) as follows:

Let \(V(G) = F_{q^t}\).

For \(a \in F_{q^t}\), let \(N(a)\) denote the \(F_{q^t}/F_q\)-norm of \(a\). i.e. \(N(a) = a \cdot a^q \cdots a^{q^{t-1}} = a^{(q^t - 1)/(q-1)} \in F_q\).

Let \(a \neq b \in V(G)\) be adjacent if and only if \(N(a + b) = 1\).

Fact: The number of solutions in \(F_{q^t}\) of the equation \(N(a) = 1\) is \((q^t - 1)/(q-1)\).

Thus, let \(n = q^t = v(G)\), then \(e(G) \geq \frac{1}{2}q^t\left(\frac{q^t - 1}{q-1}\right) - 1 \geq \frac{1}{2}q2^{t-1} = \frac{1}{2}n^{2-\frac{t}{2}}\).

Theorem 3: The graph \(G = G(q,t)\) contains no subgraph isomorphic to \(K_{t,t+1}\).

Corollary 1: For \(t \geq 2\) and \(s \geq t + 1\), we have \(e(x(n, K_{t,s})) \geq c_t \cdot n^{2-\frac{t}{2}}\), where \(c_t > 0\) is a constant depending on \(t\), we may choose \(c_t = 2^{-t}\). For every \(t\) and \(s \geq t\), the inequality holds with \(c = \frac{1}{2}\) for infinitely many values of \(n\).

Proof of Theorem 3: If \(d_1, \ldots, d_t\) are \(t\) distinct elements from \(F_{q^t}\), and we choose \(K = F_{q^t}, a_{ij} = -d_j^{i-1}, x_j = x^{a_{i1} - 1}, b_j = 1\) in Lemma 3, then the system
of equations

\[ N(x + d_1) + (x + d_1)(x^q + d_1^q) \cdots (x^{q^{t-1}} + d_1^{q^{t-1}}) = 1 \]

\[ \vdots \]

\[ N(x + d_t) + (x + d_t)(x^q + d_t^q) \cdots (x^{q^{t-1}} + d_t^{q^{t-1}}) = 1 \]

has at most \( t! \) solutions \( x \in \mathbb{F}_{q^t} \). \qed

Then, with these techniques, Alon et al. gave an improved construction.

**Definition (the general projective norm-graphs):** For any \( t > 2 \), define \( H = H_{q,t} \) as follows:

Let \( V(H) = \mathbb{F}_{q^{t-1}} \times \mathbb{F}^*_q \).

Two distinct \((A, a)\) and \((B, b)\) \( \in V(H) \) are adjacent if and only if \( N(A + B) = ab \), where \( N(X) = X^{1+q+\cdots+q^{t-2}} \).

Note that: (1) \( |V(H)| = q^t - q^{t-1} \); (2) If \((A, a)\) and \((B, b)\) are adjacent, then \((A, a)\) and \( B \neq -A \) determine \( b \).