Extremal and Probabilistic Graph Theory
Lecture 1
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First we review some Min-Max Theorems.

**Theorem 1.1** (König Theorem). *For a bipartite graph G, the minimum number of a vertex-cover in G = the maximum size of a matching in G*

**Definition 1.2.** A vertex-cover A is a set of V(G), s.t. V(G)\ A is a stable set.

**Theorem 1.3** (Menger's Theorem). *For any graph G and any subset A ,B $\subset$ V(G), the minimum size of a cut S seperating A from B in G = the maximum number of vertex-disjoint paths from A to B in G*

We will investigate some Min-Max relations on packings and coverings in digraphs.

**Definition 1.4.** A (proper) coloring of digraph D is just a (proper) coloring of its underlying graph G.

**Definition 1.5.** The chromatic number $\chi(D)$ of digraph D is the chromatic number $\chi(G)$ of graph G.

**Definition 1.6.** A path or cycle in digraph denotes a directed path or cycle.

**Theorem 1.7** (Gallai-Roy Theorem). *Every digraph D has a path with $\chi(D)$ vertices.*

*Proof.* We say a path in digraph is a u-path if the first vertex of it is u.

Let k be the number of vertices in a longest path in D, so it will sufficient to show that there exists a k-coloring of D.

Let $D^{'}$ be a maximum acyclic subdigraph of D, and for any v $\in$ V(D), define $c(v)$ to be the number of vertices in a longest v-path in $D^{'}$, then it is easy to see:$1 \leq c(v) \leq k$.

Then we show that this function $c : V(D) \to [k]$ is a proper coloring of D.

Consider any arc $(u, v) \in A(D)$, we claim that $c(u) \neq c(v)$.

Case 1:$(u, v) \in A(D)$

Consider the longest v-path P in $D^{'}$, then $u \notin P$, otherwise we get a cycle! So we have $c(u) \geq c(v) + 1$,done!

Case 2:$(u, v) \notin A(D)$

Then $(u, v) + D^{'}$ contains a cycle,for $D^{'}$ is the maximum graph.So $D^{'}$ contains a path P from v to u. Let Q denotes the longest u-path in $D^{'}$, for the similar reason, P and Q are disjoint ,except at u.Then $P \bigcup Q$ is longer than Q, so we have $c(v) > c(u)$, done! ∎

**Exercises:**

For digraph D, let $\lambda(D)$ = the number of vertices in a longest path in D. Then for graph G, we have $\chi(D) = min\{\lambda(D) :$ D is an orientation of G$\}$.

**Definition 1.8.** A path partition $\mathscr{P}$ of a digraph is a collection of vertex-disjoint paths, s.t. the union of the vertex set of those path is V(D). A single vertex can be viewed as a trivial path.

**Definition 1.9.** Let $\pi(D) = min|\mathscr{P}|$,where the minimun is taken for all path partitions, and let $\alpha(D)$=maximun size of a stable set in $D$.

Then we have:

**Theorem 1.10** (Gallai-Milgram Theorem)**.** *For any digraph D,*$\pi(D) \leq \alpha(D)$.

*Proof.* We prove a stronger statement:

We say a stable set S is orthogonal to a path partition $\mathscr{P}$, if every P $\in \mathscr{P}$ , has exactly one common vertex in S.

We also need the following lemma:

Let $\mathscr{P}$ be a path partition, suppose no stable set in D orthogonal to $\mathscr{P}$, then there exists a path partition $\mathscr{Q}$ in D, s.t. $|\mathscr{Q}| = |\mathscr{P}| - 1$, $i(\mathscr{Q}) \subset i(\mathscr{P})$, & $t(\mathscr{Q}) \subset t(\mathscr{P})$.Here, $i(\mathscr{P})$ and $t(\mathscr{P})$ denotes the sets of first vertices and last vertices in the paths of $\mathscr{P}$ respectively.

Proof of the lemma: By induction on the vertices of D.

Base case is trivial when $|V(D)| = 1$.

General cases: By the hypothesis, $t(\mathscr{P})$ is not stable. So there exists $y, z \in t(\mathscr{P})$, & $(y, z) \in A(D)$.

If $|V(P_z)| = 1$,then $\mathscr{Q}$ is obtained from $\mathscr{P}$ by deleting the path $P_z$ and replace $P_y$ by $P_y \bigcap (y, z)$, done! Here $P_y$ denotes the path with the end points y.

If $|V(P_z)| \geq 2$,let x be the predecessor of z in $P_z$. Consider $D^{'} = D\backslash\{z\}$,$P_z^{'} = P_z\backslash\{z\}$, & $\mathscr{P}^{'} = (\mathscr{P}\backslash\{P_z\}) \bigcup \{P_z^{'}\}$.

We should note that $t(P^{'}) = (t(P)\backslash\{z\}) \bigcup \{x\}$,$i(\mathscr{P}^{'}) = i(\mathscr{P})$

And if there is a stable set $S^{'}$ in $D^{'}$ orthogonal to $\mathscr{P}^{'}$, then clearly this is also a stable set in $D$ and also orthogonal to $\mathscr{P}$, which leads to a contradiction!

So, there are no such stable set in $D^{'}$. By induction on $D^{'}$, we can find a new path partition $\mathscr{Q}^{'}$ in $D^{'}$, with $|\mathscr{Q}^{'}| = |\mathscr{P}^{'}| - 1$, $i(\mathscr{Q}^{'}) \subset i(\mathscr{P}^{'})$, & $t(\mathscr{Q}^{'}) \subset t(\mathscr{P}^{'})$, then either x or y $\in t(\mathscr{Q}^{'})$.

If $x \in t(\mathscr{Q}^{'})$, add the arc $(x, z)$ is ok.Otherwise, add the arc $(y, z)$ is done!

In both cases, we have $|\mathscr{Q}| = |\mathscr{P}| - 1$, $i(\mathscr{Q}) \subset i(\mathscr{P})$, & $t(\mathscr{Q}) \subset t(\mathscr{P})$.

Now we turn to prove the main theorem:

Choose $\mathscr{P}$ to be the minimum partition of D, then by our lemma before, there must be a stable set S orthogonal to $\mathscr{P}$, so $\pi(D) = |\mathscr{P}| \leq |S| \leq \alpha(D)$. ∎

**Summary:**

Gallai-Roy Theorem says that any digraph D has a path with $\chi(D)$ vertices, or equivalently, Minimum number of disjoint stable sets whose union in $V(D) \leq$ maximum of the number of vertices in a path of $D$.

Gallai-Milgram Theorem says that for any digraph D we have $\pi(D) \leq \alpha(D)$, or equivalently, Minimum number of disjoint paths whose union in $V(D) \leq$ maximum of the number of vertices in a stable set of $D$.

Now we can see that Gallai-Roy Theorem and Gallai-Milgram Theorem have the following *dual* relation: one can be transformed from another by just interchanging the rolse of *paths* and *stable sets*.