Extremal and Probabilistic Graph Theory

June 1st, Thursday

Lemma 8. Let $\delta > 0$, $r \geq 3$ and $k \geq 4$. Let H be an r-graph and $E \subset \partial H$ with $|E| > \delta \cdot n^{r-1}$. Suppose $d_H(f) > l+1$ for $\forall f \in E$ and if $r = 3$ and k is odd, then in addition, for $\forall f = xy \in E$, there is $e_f = xy\alpha \in H$ s.t. $\min\{d_H(x\alpha), d_H(y\alpha)\} \geq 2$ and $\max\{d_H(x\alpha), d_H(y\alpha)\} \geq 3k+1$. Then for large n, $C_k \subset H$, $P_k \subset H$.
Proof. By lemma 7, it suffices to show $\psi_t(H) \neq \emptyset$.

Let T be a random subset of $V(H)$ obtained by picking each vertex of $V(H)$ with probability $p = \frac{1}{2}$. Let $F = \{f \in E | f \subset T, |N_H(f) - T| \geq m = l+1, e_f - f \notin T\}$.

For $f \in E$, fix the m edges $e_1, \cdots, e_m \in H$ containing f s,t, $e_1 = e_f$. The probability that $f \subset T$ and $e_i - f \notin T$ for $\forall i \in [m]$ is exactly $(\frac{1}{2})^{r-1+m}$, so $E[|F|] \geq |E|(\frac{1}{2})^{r+l} \geq \delta n^{r-1} 2^{-(r+l)}$, so there is a subset $T \subset V(H)$ with $|F| \geq \delta 2^{-(r+l)} n^{r-1}$.

By lemma 3, there is a complete $(r-1)$-partite $(r-1)$-graph $G \subset F$ with each part of size t. Since $|L_G(f)| \geq |N_H(f) - T| \geq l+1$ for $\forall f \in G$, we prove that $\psi_t(H) \neq \emptyset$.

Recall
Def. An n-vertex r-graph H is (t,c)-sparse if $\forall$ t-subset of vertices lies in at most c edges of H.
Lemma 4. Fix $c > -$ and $r, k \geq 3$. Let H be an n-vertex $(r-1,c)$-sparse r-graph not containing P_k. Then $|H| = o(n^{r-3})$.

Main Thm(Asymptotics). Let $r \geq 3$, $k \geq 4$.
(a) If H is an n-vertex $(l+1)$-full r-graph and H is C_k-free or P_k-free. Then $|H| = o(n^{r-1})$.
(b) $ex_r(n,P_k) \sim ex_r(n,C_k) \sim l \cdot \binom{n}{r-1}$.
Proof. (a) We claim $|\partial H| = o(n^{r-1})$.

Suppose not that $|\partial H| \geq \delta \cdot n^{r-1}$ where $\delta > 0$ and n is large.

If $r \geq 4$ or $r = 3$&k is even, by lemma 8, H contains C_k and P_k, a contradiction.

So assume r=3 and k is odd. Let H^* be the set of edges of H containg NO pairs of codegree at least $3k$. So H^* is $(2,3k)$-sparse. By lemma4, $|H^*| = o(n^2)$.

Let $F = \partial H - \partial H^*$. So for $\forall f \in F$, there is an $e \in H$ containing f and containing a pair $f^{'}$ with $d_H(f^{'}) \geq 3k+1$. Then $|F| \geq |\partial H| - |\partial H^*| \geq \delta \cdot n^2 - o(n^2) \geq \frac{\delta}{2} n^2$ (for large n).

Define $E \subset F$ to be the set as in lemma 8. Fix $f \in F$, we want to map f to some $f^{'} \in E$.

Case 1. If all edges, say $xyz \in H$, containing f, satisfy that $d_H(xy) \geq 3k+1$, $d_H(xz) \geq 3k+1$, $d_H(yz) \geq 3k+1$, we can just map f to itself.

Case 2. $\exists$ an edge $xyz \in H$ containing f s.t. one of the pairs $f^{'}$ has codegree at most $3k$ and one of the pairs has codegree at least $3k+1$, we can map f to $f^{'} \in E$.

Claim. There are at most $6k+1$ subedges $f \in F$ which can be mapped to the same $f^{'} \in E$.

If f is in Case 1, then it is 1 to 1 mapping.

If f is in Case 2, then $d_H(f^{'}) \leq 3k$, therefore there are at most $6k+1$ pairs can lie in an edge containing $f^{'}$

This claim shows $|E| \geq |F|/(6k+1) \geq (\delta/(12k+2)) \cdot n^2$. Then by lemma 8, we can find C_k and P_k. This proves the claim that $|\partial H| \leq o(n^{r-1})$.

By lemma 1, H has a $r(k+1)$-full subgraph $H^{'}$ with $|H^{'}| \geq |H| - r(k+1)|\partial H|$.

By lemma 2, if $H^{'} \neq \emptyset$, then $H^{'} \supset C_k$, a contradiction.

Then $H^{'} = \emptyset \Rightarrow |H| \leq r(k+1)|\partial H| = o(n^{r-1})$.