Extremal and Probabilistic Graph Theory
Lecture 2
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Notation

In this lecture, we use G denotes a graph, and D denotes a digraph.
A k-graph means a k-uniform hypergraph.

Recall

Gallai-Roy Theorem

For any digraph D, there's a directed path with at least $\chi(D)$ vertices.

Gallai-Milgram Theorem

For any digraph D, we have $\pi(D) \leq \alpha(D)$.

Definition 2.1. A partial ordered set (or poset) is an ordered pair (X, R),where R is a relation on X and usually replaced by $\preceq$, satisfying the following three properties:

- reflexive:$x \preceq x, \forall x \in X$
- antisymmetric: if $x \preceq y$ and $y \preceq x$,then we have $x = y$
- transitive:if $x \preceq y$ and $y \preceq z$,then we have $x \preceq z$

An antichain means that any two elements $x, y \in X$,satisfying neither $x \preceq y$ nor $y \preceq x$.

Theorem 2.2 (Dilworth's Theorem). *For any poset $P = (X, \preceq)$, the minimum number of disjoint chains whose union is X = the maximum number of elements in antichain.*

Proof. The proof of Dilworth's Theorem is left as an exercise, which can be done via Gallai-Milgram Theorem. ∎

For the relation between cycles and $\alpha(D)$ we cannot hope to partition $V(D)$ into cycles. In this lecture, we should consider cycle covering.

Theorem 2.3 (Bessy-Thomasse Theorem/Gallai's Conjecture). *For strongly connected digraph D, $V(D)$ can be covered by at most $\alpha(D)$ cycles, i.e. there exists cycles $C_1, C_2 ... C_\alpha$, s.t. $\bigcup_{i=1}^{\alpha} V(C_i) = V(D)$.*

Proof. The proof of Bessy-Thomasse Theorem is not difficult and we also leave it as an exercise. Considering the ear-composition may be helpful. ∎

But for graphs G, it is possible to extend Gallai-Milgram Theorem by partition $V(G)$ into $\alpha(D)$ cycles (instead of paths).

Theorem 2.4 (Pósa's Theorem). *For any graph G, $V(G)$ can be partitioned into at most $\alpha(G)$ disjoint cycles, edges and vertices.*

Proof. We prove by induction on $\alpha(G)$.

Because when $\alpha(G) = 1$, then $G = K_n$ is complete, the conclusion is trivial.

Now consider $\alpha(G) \geq 2$.

If there is a vertex v with $d_G(v) = 0$, then as $\alpha(G - v) = \alpha(G) - 1$ (for v is an isolated point), we can use induction on $G - v$ to get that $V(G) - v$ can be partitioned into at most $\alpha(G) - 1$ cycles, vertices and edges. Then adding v back, this partition becomes a desired partition of G.

If there is a vertex v with $d_G(v) = 1$, let $N_G(v) = \{u\}$, consider $G - \{u, v\}$.

Claim 1:$\alpha(G - \{u, v\}) \leq \alpha(G) - 1$.

Proof of claim1: This is ture because that any stable set A in $G - \{u, v\}$ will give a stable set $A \cup \{v\}$ in G. Take A to be maximum, then we have $\alpha(G) \geq |A \cup \{v\}| = |A| + 1 = \alpha(G - \{u, v\}) + 1$.

By induction on $G - \{u, v\}$ to get that $V(G - \{u, v\})$ can be partitioned into at most $\alpha(G - \{u, v\}) - 1$ cycles, vertices and edges. Then adding the edge (u, v) back, this partition becomes a desired partition of G.

Therefore, we can assume $\delta(G) \geq 2$.

Claim 2: There exists a cycle C and a vertex $v \in V(C)$, s.t. $N_G(v) \subset V(C)$.

Proof of claim2: Take a longest path P in G with two endpoints $a\&b$. Then $N_G(a) \subset V(P)$ for P is the longest path in G. Take the furthest neighbour w of a on P, then we have a cycle $C = aPw \cup \{aw\}$.

Now consider $G - V(C)$.

Claim 3:$\alpha(G - V(C)) \leq \alpha(G) - 1$.

Proof of claim3: This is ture because that any stable set A in $G - V(C)$ will give a stable set $A \cup \{v\}$ in G. Take A to be maximum, then we have $\alpha(G) \geq |A \cup \{v\}| = |A| + 1 = \alpha(G - V(C)) + 1$.

By induction on $G - V(C)$ to get that $G - V(C)$ can be partitioned into at most $\alpha(G - V(C)) - 1$ cycles, vertices and edges. Then adding the cycle C back, this partition becomes a desired partition of G.

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Definition 2.5. For a hypergraph H, a stable set S is a subset of $V(H)$ which spans no hypergraphs. Then we use $\alpha(H)$ denotes the maximum of $|S|$, where the maximum is taken from all stable sets S.

Definition 2.6. A linear cycle in a hypergraph H is a sequences of hyperedges $e_1, e_2...e_r$, s.t.

for any $i < j$, $|e_i \cap e_j| = \begin{cases} 1, & \text{if } \ j = i + 1 \ \text{ or } \ (i, j) = (1, r); \\ 0, & \text{otherwise.} \end{cases}$

Theorem 2.7 (Gyárfás-Sarközy Conjecture(Open))**.** *For any k-graph H, $V(H)$ can be partitioned into at most $\alpha(H)$ linear cycles ,vertex and subsets of hyperedges.*

Remark

To see why we consider *subsets of hyperedges* instead of *hyperedges*, $K_5^{(3)}$ (the complete 3-graph) is just an example.

Exercise

If is easy to prove the conjecture by replacing *linear cycles* to *linear paths*, and this is left as an exercises.

Definition 2.8. A weak cycle in a hypergraph H is a sequences of hyperedges $e_1, e_2...e_r$, s.t.

for any $i < j$, $e_i \cap e_j \begin{cases} \neq \emptyset, & \text{if } \ j = i + 1 \ \text{ or } \ (i, j) = (1, r); \\ = \emptyset, & \text{otherwise.} \end{cases}$

Theorem 2.9 (Gyárfás-Sarközy)**.** *For any k-graph H, $V(H)$ can be partitioned into at most $\alpha(H)$ weak cycles ,vertex and subsets of hyperedges.*

Remark

Recall the proof the *$P\acute{o}sa's$ $Theorem$* we should notice that the KEY is to find a suitable part C and show that

$$\alpha(G - V(C)) \leq \alpha(G) - 1$$

Proof. By induction on $\alpha(H)$.

Base case when $\alpha(H) = 1$ is trivial, because when this happens it means that H has no edges and has only one vertex.

So we consider $\alpha(H) \geq 2$.

If H has no edges which means that $\alpha(H) = |V(H)|$, which is also trivial. So we can assume that H has at least 1 edge.

Take a maximal weak path P in H .Let $V(P) = \{1, 2, ..., s\}$, $E(P) = \{e_1, e_2, ..., e_t\}$, $M = e_{t-1} \cap e_t$, $Y = V(P) - e_t$,and $F_1 = \{e \in E(H) : e \cap M \neq \emptyset, s \in e\}$.

Claim 1: $F_1 = \emptyset$.

Suppose that $F_1 \neq \emptyset$, then clearly that $\forall f \in F_1$ must intersect with Y for P is maximal. For each $f \in F_1$, let $x_f \in f \cap Y$ be the maximum integer, and choose $f^* \in F_1$ s.t. x_{f^*} is the minimum among the x_f's. We also let g be the unique edge of P containing x_{f^*} and closer to S.

Let C be the weak cycle formed by edges $(f^*, g, ..., e_t)$,we will show that $\alpha(H - V(C)) \leq \alpha(H) - 1$.

Claim that for any stable set A in $H - V(C)$, we see $A \cup \{s\}$ is stable in H. If there exists $f_0 \subset A \cup \{s\}$, then $f_0 \in F_1$ and $f_0 \cap (V(P) - V(C)) = \emptyset$, which is a contradiction to the minimality of x_{f^*}.

Then by induction on $H - V(C)$ to get that $H - V(C)$ can be partitioned into at most $\alpha(H - V(C)) - 1$ weak cycles, vertices and subsets of hyperedges. Then adding the weak cycle C back, this partition becomes a desired partition of H, and claim 1 is proved.

Let $F_2 = \{e \in E(H) : s \in e, e \neq e_t\}$.

Claim 2: $F_2 = \emptyset$.

Suppose that $F_2 \neq \emptyset$, then clearly that $\forall f \in F_2$ has $f \cap M \neq \emptyset$. Let $C = e_t \cup f$ be a weak cycle on 2 edges. Then we show again that $\alpha(H - V(C)) \leq \alpha(H) - 1$.

Because any stable set A in $H - V(C)$ gives a stable set $A \cup \{s\}$ in H. If there exists $f_0 \subset A \cup \{s\}$, then $f_0 \in F_1$ while $F_1 = \emptyset$, which also leads to a contradiction!

Then by induction on $H - V(C)$ to get that $H - V(C)$ can be partitioned into at most $\alpha(H - V(C)) - 1$ weak cycles, vertices and subsets of hyperedges. Then adding the weak cycle C back, this partition becomes a desired partition of H, and claim 2 is proved.

Therefore, the only edge in H containing s is e_t, and it is obviously right that $\alpha(H - e_t) \leq \alpha(H) - 1$. Then by induction on $H - e_t$ to get that $H - e_t$ can be partitioned into at most $\alpha(H - e_t) - 1$ weak cycles, vertices and subsets of hyperedges. Then adding the hyperedge e_t back, this partition becomes a desired partition of H. Done!

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