Extremal and Probabilistic Graph Theory
Lecture 3
March 7st, Tuesday

- **Definition 1.** let $\rho(H)$ be the minimum number of edges or subsets of edges that are needed to partition $V(H)$.

- **Definition 2.** let $\chi(H)$ be the chromatic number of H, i.e, the minimum k such that $V(H)$ can be partitioned into k stable sets.

- **Remark:** (1) This confirms the G-S Conjecture for 3-graph without linear cycles.

  (2) Consider $H = K_5^{(3)} \Rightarrow$ tight.

- **Definition 3.** A linear tree is a 3-graph that is obtained from a single edge by repeating adding edges that intersect the previous hypergraph in exactly one vertex.

- **Definition 4.** A skeleton T of H is a linear subtree of H which can not be extended to a larger linear subtree by adding new edges.

- **Definition 5.** For a hypergraph H, the underlying graph ∂H of H is a graph with the same vertex set $V(H)$ and $E(\partial H) = \{(x, y)|\{x, y\} \subseteq e \in E(H)\}$. For a vertex $v \in V(T)$, where T is a linear tree in H, the pairs (a, b) that are at equal distance from v in the underlying graph of T and belong to one edge of T has exactly one opposite pair to v.

- **Definition 6.** Let $\theta(G)$ be the minimum number of complete subgraphs in G which vertices cover $V(G)$.

- **Lemma 1.**Let H be a 3-graph without linear cycle, and T be a linear tree of H. let $v \in V(T)$, if $f = (v, a, b) \in E(H)$ such that $\{a, b\} \cap V(T) \neq \emptyset$. Then either $\{a, b\}$ intersects with some edges of $E(T)$ which contains v, or $\{a, b\}$ is an opposite pair to v in T.

  Proof: Omit.

- **Lemma 2.** Let T be a linear tree and G be its underlying graph. Then $\alpha(G) = \theta(G)$.

  Proof: Exercise.

- **Theorem 1**(Gyárfás-Gyori-Simonovits,2016). If H is 3-graph without linear cycles, then $\rho(H) \leq \chi(H)$ and moreover, $\chi(H) \leq 3$.

  Proof. Choose a skeleton T_1 in it, then choose a skeleton T_2 in $H \backslash T_1$, and continue this process to get $T_3, \cdots, T_m$, until an edgeless T_{m+1}.For each $1 \leq i \leq m+1$, let G_i be the underlying graph of T_i. By lemma 2, $\alpha(G_i) = \theta(G_i)$.

  Fact 1 : By the maximality of skeletons, for any j, there are no edges of H intersecting $V(T_j)$ in one vertex and intersecting $\bigcup_{i>j} V(T_i)$ in two vertices.

  Fact 2 : Consider a stable set $S \subseteq V(G_j)$. As H has no linear cycle, by lemma 1, no edge of H is in S and no edge of H contains exactly two vertices in S.

  Claim: For each stable set $S_i \subseteq V(G_i), \bigcup_i S_i$ is a stable set in H.

Proof. Following from fact 1 and 2. ∎

Therefore, $\alpha(H) \geq \sum_{i=1}^{m+1} \alpha(G_i) = \sum_{i=1}^{m+1} \theta(G_i) = \sum_{i=1}^{m+1} \rho(T_i) \geq \rho(H)$. The second assertion $\chi(H) \leq 3$ also follows by the fact $\chi(G_i) \leq 3$ and the claim. ∎

- **Theorem 2.**(Ergemlidze-Györi-Methvku)If H is a 3-graph without linear cycles, then either $H \geq K_5^{(3)}$ or $\chi(H) \leq 2$.
- **Theorem 3.**(Ergemlidze-Györi-Methvku)For 3-graph H, $V(H)$ can be covered by at most $\alpha(H)$ linear cycles, vertices or edges.