Extremal and Probabilistic Graph Theory
Lecture 4
March 14st, Tuesday

Recall the lecture in the last lesson:

- **Lemma.**Let H be a 3-graph without linear cycle, and $T$ be a linear tree of $H$. Let $v \in V(T)$. If $f = (v, a, b) \in E(H)$ is such that $\{a, b\} \cap V(T) \neq \emptyset$, then either $\{a, b\}$ intersects with some edges of $E(T)$ which contains $v$, or $\{a, b\}$ is an opposite pair to $v$ in $T$.

- **G-S Conjecture.** Any $k$-graph $H$ can be partitioned into at most $\alpha(H)$ linear cycles, vertices or subsets of hyperedges.

- **Theorem(G-S).** This is true if replacing "linear cycles" with "weak cycles".

- **Theorem(Ergemlidze-Györi-Methuku,2017).** For 3-graph $H$, $V(H)$ Can be covered at most $\alpha(H)$ linear cycles, vertices and hyperedges.

- **Definition.** A hypergraph $H$ is mixed, if the size of any hyperedge is either 2 or 3.

  The concept of linear cycles can be easily extended to mixed hypergraph.

- **Theorem(E-G-M).** For any mixed hypergraph $H$, $V(H)$ can be covered by $\alpha(H)$ linear cycles, vertices and hyperedges.

  **Proof:** By induction on $\alpha(H)$.

  Base case:$\alpha(H) = 1$, then $H$ contains a complete graph, which gives a Hamilton cycle.Done. Let us assume the statement holds for any mixed hypergraph $H'$ with $\alpha(H) < \alpha(H')$. Consider $H$, Then we may assume $E(H) \neq \phi$. Let $P$ be the longest linear path in $H$ consisting of hyperedges $h_0, h_1, ..., h_l$. If $h_i$ is of size 3, write $h_i = v_i v_{i+1} u_{i+1}$, and if $h_i$ is of size 2, write $h_i = v_i v_{i+1}$. An initial segment of $P$ consisting of $h_0, h_1, ..., h_i$. Let $C$ be a linear cycle in $H$ which contains the longest initial segment of $P$. If there is no linear cycle containing $h_0$, then we let $C = h_0$.

  An initial segment of $P$ is a linear subpath of $P$ consisting of $h_0, h_1, ..., h_i$. Let $R = \{v_k u_k | \{u_k, v_k\} \subseteq V(P) - V(C)$, and $v_0 v_k u_k \in E(H)\}$. Let $H'$ be obtained from $H - V(C)$ by adding the pairs $v_k u_k \in R$ as new edges.

  Claim 1:$\alpha(H') \leq \alpha(H) - 1$.

  **Proof:**We in fact can show: for any stable set $I$ in $H'$, $I \bigcup \{v_0\}$ is also a stable set in $H$. Suppose by contradiction that there is an edge $h \subseteq I \bigcup \{v_0\}$ in $H$. Clearly $v_0 \in h$, next, we consider three cases below:

  Case 1: $|h \bigcap (V(P) - V(C))| = 0$.

  This case is wrong, otherwise, we can find a longest linear path than $P$.

  Case 2: $|h \bigcap (V(P) - V(C))| = 1$.

  In this case, we can find a linear cycle $\widetilde{C}$ which contains a longest initial segment than $C$, a contradiction.

  Case 3: $|h \bigcap (V(P) - V(C))| = 2$.

By the proof of “that lemma”, $h\bigcap(V(P)-V(C))$ must be an opposite pair of $P$, say $v_ku_k\in R$. But $v_ku_k$ is also belongs to $I$, a contradiction. ∎

Claim 2: The set of hyperedges of any linear cycle in $H'$ can contain at most one new edge $v_ku_k$ from $R$.

**Proof:** Suppose a linear $D$ in $H'$ contains 2 pairs from $R$. Then, we can find a linear subpath $P'$ of $D$ consisting of edges $h'_0, h'_1, ..., h'_m$, where $h'_0=v_su_s, h'_m=v_tu_t$ for $s<t$.

Let $h_i$ be the edge of $P$ which intersects with $P'$, and subject to this, $i$ is minimum.

Let $h'_j$ be $h'_j\bigcap h_i\neq\phi$ and subject to this $j$ is minimum.

We consider two cases below:

Case 1: $|h'_j\bigcap h_i|=1$. Then we can find a new linear cycle $h_0, h_1, ..., h_i, h'_j, h'_{j-1}, ..., h'_1, v_0v_su_s$, which contains a longest initial segment than $C$.

Case 2: $|h'_j\bigcap h_i|=2$. Then $|h'_{j+1}\bigcap h_i|=1$., so we have a linear cycle $h_0, h_1, ..., h_i, h'_{j+1}, h'_{j+2}, ..., h'_{m-1}, v_0v_tu_t$, which also contains a longer initial segment. This is a contradiction. ∎

So claim 2 shows that any linear cycle $D'$ in $H'$ can be extended to a linear cycle $D$ in $H$, such that $V(D')\subseteq V(D)\subseteq V(D')\bigcup\{v_0\}$. By induction on $H'$, $V(H')=V(H)-V(C)$. And $V(H')$ can be covered by at most $\alpha(H')\leq\alpha(H)-1$ linear cycle of $H'$, vertices and hyperedges. In view of the above observation, together with the linear cycle $C$ is the desired covering of $H$. ∎